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ON RELATIVE VELOCITY.\*

BY H. E. PIGGOTT.

To the observant and intelligent boy or girl who has ever sailed on the Broads or steered a motor-boat in a busy harbour, the ordinary text-book treatment of Relative Velocity must be unsatisfying. The velocity of a ship (or other moving body)  $P$  relative to a ship  $Q$  is defined as the velocity of  $P$  as it "appears" to an observer on  $Q$ . But the exact meaning of "appears" is seldom explained. The fact that in all practical applications, observations are being made from one or other of the moving platforms, not from a stationary airship above them, is generally obscured. Thus the relative velocity of  $P$  with respect to  $Q$  is presented as something mathematical, obtained by the trick of reversing  $Q$ 's velocity and adding it vectorially to that of  $P$ . Now, anyone who has steered a boat and observed another approaching on a converging course, knows that he is not directly concerned with the "actual" velocities (meaning velocities relative to the earth) of either boat. On the other hand, the velocity relative to him of the other boat is a very real thing. But to obtain it he does not have to go through the mental process of imagining that he is going astern when he knows that he is going ahead. All he does is to look over the side and observe. If the other boat's relative bearing does not change, or if its rate of change is small, he knows that it is time to do something with the tiller.

If from the deck of a ship  $Q$  at sea we are watching a distant ship  $P$ , the latter is not, as a rule, considerate enough to signal her course and speed, nor have we any means of discovering these by direct observation.

If, however, both ships are proceeding with constant speeds and

\* The substance of a talk given before the Mathematical Association at the Annual Meeting, 6th January, 1933.

These remarks were suggested in the first place by Mr. D. A. Young's Note, 1096, *Math. Gazette*, XVI, October, 1932.

on constant courses, five observations are sufficient to determine her velocity relative to us, namely, two sets of simultaneous ranges and compass-bearings and the time-interval between them.

If these two sets of observations are plotted by means of two rays  $Q_1P_1$  and  $Q_1P_2$  (see Fig. 1),  $P_1P_2$  gives the track of  $P$  relative to  $Q$ . By measurement (or calculation) of  $P_1P_2$  and by using the observed time-interval we obtain the speed of  $P$  relative to  $Q$ . From this

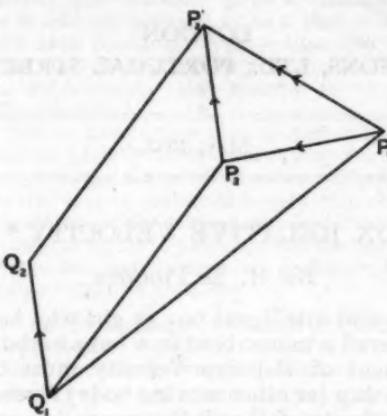


FIG. 1.

construction alone, a number of predictions which may be of considerable importance may be made—for instance, how nearly the ships will approach, the instant of this shortest range, when  $P$  will be on a given bearing from  $Q$  and what will be her range at that instant, and so on.

Suppose, however, we require the "actual" speed and course of  $P$ . From a knowledge of the speed and course of our own ship, we can plot our run  $Q_1Q_2$  between the instants when the observations were taken.

The figure may now be regarded as drawn on a chart,  $Q_1$  and  $Q_2$  being the first and second positions of  $Q$ , and  $P_1$  the first position of  $P$ . To obtain the second position of  $P$ , we should have plotted the ray representing the second set of observations from  $Q_2$  instead of from  $Q_1$ .  $Q_1P_2P_2Q_2$  is thus a parallelogram, and  $P_1P_2$  gives the charted track of  $P$ . From this we deduce  $P$ 's "actual" course and speed.

But note the vector triangle  $P_1P_2P_2$ . This gives us the rule: "Compound the velocity (relative to the earth) of  $Q$  with the velocity of  $P$  relative to  $Q$ . The resultant is the velocity of  $P$  (relative to the earth)." This is the same as the rule given by Mr. Young, viz.  $V_Q + V_{P-Q} = V_P$ , but it is here obtained from elementary considerations, based on a definite conception of the meaning of relative velocity, not deduced from a rule which employs a fictitious velocity.

Many problems of interest and practical importance can be solved by direct or indirect use of this rule. Thus : "A is a ship steaming due north at 12 knots. From a second ship B, A bears N.  $35^\circ$  E., distant 2 nautical miles. B can steam at 16 knots. What course should B steer so as to come up with A in the shortest time, and after how long should she be alongside ? "

In the velocity-triangle  $PQR$  (Fig. 2) we have  $PQ$  representing



FIG. 2.

$A$ 's velocity (12 knots, due N.).  $RP$  the direction of the velocity of  $B$  relative to  $A$  (so that angle  $RPQ = 180^\circ - 35^\circ$ ), and the length of  $QR$ , representing  $B$ 's speed, 16 knots. Thus the triangle can be constructed by an arc of a circle, and the results required measured from the figure or calculated by sine rule. We find that angle  $RQP = 9\frac{1}{2}^\circ$ , and the relative speed  $RP$  is 4.6 knots, whence the time-interval required is 26 minutes.

Now, a considerable proportion of exercises on Relative Velocity found in Mechanics text-books and in examination papers visualise the problems, not primarily from the point of view of observers on the moving platforms but from that of "the little cherub up aloft". That is to say, they require a relative velocity to be obtained from two "actual" velocities. As an illustration, we may take the London Matriculation question quoted by Mr. Young : "A steamship is travelling north at a rate of 10 miles an hour, and there is a north-east wind blowing at the rate of 20 miles an hour. In what direction will the smoke from the funnel appear to move to an observer on the ship ? " This is, of course, a perfectly fair question, and no idea of relative velocity need enter into its solution. It can be done by considering the positions after any interval (say 1 hour) of the smoke-particles which have been emitted from the funnel during that interval. But, for all that, it is difficult to imagine that the examiner who framed the wording of the question had ever stood on the deck of a steamship and had watched, with active brain, the smoke trailing out at an angle with the fore-and-aft line (or that he had observed and meditated on the changing angle of the splash-lines made by falling raindrops on the window-pane of a railway carriage of a train gathering speed). The natural form of the

question would be to give the steamship's speed and course, the angle between the smoke-track and the fore-and-aft line, with either the wind's speed or its direction, and to require the other. (If the speed is given the solution may be ambiguous.) It makes the question more satisfactory, but slightly harder, if the ship is made to change its course and a second angle be given. Then both speed and direction of the wind can be determined. This problem is tiresome by calculation, but the geometrical solution is neat. Suppose that when the ship is steaming due north, the smoke-track makes  $30^\circ$  with the fore-and-aft line, on the eastern side, and when the course is changed to N.  $50^\circ$  E. the smoke-track angle is  $60^\circ$ .  $OP$  and  $OQ$  are each drawn of the scale-length of the ship's speed (10 knots),  $OP$  being N., and  $OQ$ , N.  $50^\circ$  E. Draw  $OPR$   $30^\circ$  and  $OQR$   $60^\circ$  as shown (Fig. 3). Then  $OR$  gives the speed and direction of the wind.

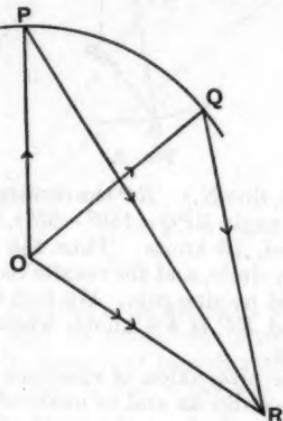


FIG. 3.

For in each of the vector triangles,  $OPR$ ,  $OQR$ , we have ship's velocity, compounded with smoke-velocity (really velocity of wind relative to ship) gives wind velocity. This is the method commonly used by air-pilots to determine the velocity of the wind. The smoke-track angle is replaced by the angle of drift of a landmark observed on the ground vertically below the aircraft.

The following is another problem illustrating the same principle:  $ABC$  is a triangular course for aircraft,  $A$ ,  $B$ , and  $C$  being three landmarks. An airplane flies round the course with constant air-speed, but, owing to the wind, the ground-speeds over the portions  $AB$ ,  $BC$ ,  $CA$ , respectively, are 60, 50, 90 miles per hour. Find the speed and direction of the wind.

From a point  $O$ , draw  $OP$ ,  $OQ$ ,  $OR$ , respectively, parallel to  $AB$ ,  $BC$ ,  $CA$ , and of scale-lengths to represent 60, 50, 90. Find  $O'$ , the circum-centre of the triangle  $PQR$ . Then  $OO'$  gives the speed and direction of the wind. For in each of the triangles  $OO'P$ ,  $OO'Q$ ,  $OO'R$ , we have

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wind-velocity, compounded with air-velocity gives ground-velocity, and since  $O'P$ ,  $O'Q$ ,  $O'R$  are all equal, the air-speed is constant.

It is not suggested that the form of question in which a relative velocity has to be determined from two "actual" (i.e. "earth") velocities should be tabooed. The problem arises sometimes in practice and it affords a reasonable test of the principles of this branch of Mechanics. But any student who has grasped Relative Velocity as a real conception will have no difficulty here. Nor is there any need for a restatement of the main rule. It is perfectly easy to build up the vector triangle from any sufficient data. Moreover, it is more natural to subtract the vectors by the method of "complementary addition" ( $V_{P-Q}$  is that vector which added to  $V_Q$  gives  $V_P$ ) than to reverse  $Q$ 's velocity and add this to  $P$ 's.

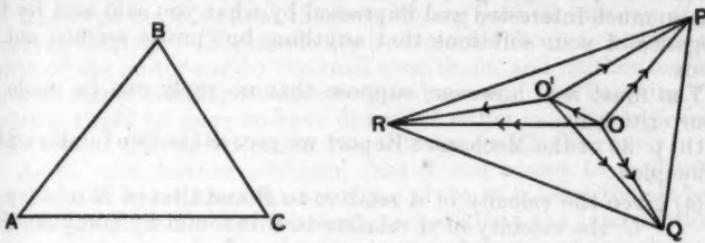


FIG. 4.

In the M.A. Report on the Teaching of Mechanics in Schools (1930), on pages 46 and 47 it is recommended that fictitious forces (that is, accelerations) be avoided. It would have been more logical if the compilers of that Report had also avoided recommending the use of a fictitious velocity on page 39. On that page are stated two fundamental principles. The first is: "Given the velocity of  $A$  relative to  $B$ , and that of  $B$  relative to  $C$ , the velocity of  $A$  relative to  $C$  is found by compounding the two velocities." (Why not have stopped here, with, possibly, a note that in practical problems  $C$  is often the earth and that such velocities are commonly, though incorrectly, referred to as 'actual,' or without any qualifying adjective?) Next we have, quite unnecessarily, and stated as a second fundamental principle, "Given the velocities of  $A$  and  $B$  each relative to  $X$ , the velocity of  $A$  relative to  $B$  is found by compounding  $A$ 's velocity with the reverse of  $B$ 's."

Why should a fictitious velocity be commendable, and a fictitious acceleration be reprehensible? Is the operation of differentiation so devastating when applied to a time-function? Also, why state two fundamental principles, when the second is implicit in the first? Anyway, what about Occam's razor?

**The President:** I think the trouble which has led to Mr. Piggott writing his paper is possibly due to the fact that geometry and algebra have received much more attention in the last thirty or forty years and we now have reasonable text-books on those subjects, whereas the writers of text-books on statics and dynamics have not progressed

at the same rate, and the writers of the text-books, followed, I regret to see, by the Committee that produced the Mechanics Report, have rather tended to reproduce a meaningless formula without considering the full meaning of what they are really saying.

I will offer the thanks of the Association to Mr. Piggott for his interesting paper.

A REPLY BY MR. C. O. TUCKEY.

DEAR MR. PIGGOTT,

No reply was attempted by me to your criticism of the Report on Mechanics at the Mathematical Association meeting partly because of the doubt whether on the spur of the moment I could have put my argument in its best form, but largely because I was so much interested and impressed by what you said and by the elegance of your solutions that anything but praise seemed out of place.

You must not, however, suppose that no reply can be made to your criticisms.

On p. 39 of the Mechanics Report we give as the two fundamental principles :

- (a) given the velocity of  $A$  relative to  $B$  and that of  $B$  relative to  $C$ , the velocity of  $A$  relative to  $C$  is found by compounding the two velocities ;
- (b) given the velocities of  $A$  and  $B$  each relative to  $X$ , the velocity of  $A$  relative to  $B$  is found by compounding  $A$ 's velocity with the reverse of  $B$ 's.

You pointed out that for the man on the ship (a) was the important one ; that all the important practical problems of navigation were solved from it ; and you suggested that it was only because the Teaching Committee failed to take the point of view of the man in the ship that they stated (b) as if of equal importance.

For this part of your criticism you certainly made out a good case, so that perhaps it is well to mention that we nowhere said that the principles were of equal importance. Still, it is very likely that the Committee, who after all were on land, did not emphasise the point of view of the sailor sufficiently ; but surely your navigators, as they steam away from the land, might sympathise a little with the man left behind on the pier and grant that it is a legitimate aspiration for him to try to put himself in their place and by means of principle (b) to see with their eyes. Besides, might it not be of interest even to your observer on board the battleship to decide the relative motion of two attendant destroyers, or whether a friendly ship would need to alter its course to avoid too near an approach to a hostile one ?

If, however, you merely had charged the Committee with having suggested that two principles were of equal importance instead of emphasising the greater importance of the first, it might have been better not to have replied ; we might well agree to differ on a question of emphasis.

Your more unkind criticism, however, was that we ought not to

have introduced a *fictitious* velocity (the "reverse of *B*'s") when elsewhere in the Report we condemn the use of fictitious forces.

This I would meet by saying that we do *not* introduce a fictitious velocity.

Of course, in a sense, all observed velocities have a fictitious element, as they all depend on the motion of the point of observation, but I suggest that you, in your capacity of observer on the deck of a ship *B*, have no right to speak of the observed velocity of a ship *A* as *actual* and then stigmatise as *fictitious* the observed velocity of the water on which the ship *A* rests.

Perhaps you find it hard to adapt your rangefinder and other instruments to observe accurately the velocity of the water, but surely this does not make the velocity fictitious. Besides, you could observe it quite easily if the ship was passing a rock.

Have you never noticed the trees flitting past the window of your railway carriage? Their velocity is no more difficult to observe than that of the motor-car on the road near them, and neither more nor less an *actual* velocity relative to the observer. The Teaching Committee would be sorry to have described either velocity as *fictitious*, and I hope that on reflection you will agree with their view.

As to your further criticism, that if one wishes to deal with a problem of which the data are those of (b), it is nevertheless better to use the principle (a) backwards and so avoid the introduction of the second principle (b); this is a matter of classroom experience and it will not be surprising if opinions differ. My own experience tends to show that it is easier to use two principles each in a single straightforward way rather than to employ only one, but to use it in two different ways.

It is rather like the question whether one should use the table of anti-logarithms as well as that of logarithms, or whether it is better to use the table of logarithms backwards. A great many teachers advocate the latter procedure, but none, I think, on the ground that it is *easier*. They say that to use the logarithms only is more accurate and *not much harder*.

In the case of relative velocities, the two suggested methods do not differ in accuracy; and if the analogy has any force, it would appear that the straightforward use of the two principles is the easier method and that the recommendation in the Report on the Teaching of Mechanics is justified.

Yours sincerely,

C. O. TUCKEY  
(Chairman of the Teaching Committee).

#### GLEANINGS FAR AND NEAR.

906. A biological professor cannot possibly mean that a dog has no sagacity or intelligence of his own. Nor can the dog lover claim that his hound could ever do crossword puzzles or solve "a simultaneous equation with three unknowns"—that, according to a speaker at the British Association, being the proper qualification for a member of Parliament.—A letter to *The Times*, Sept. 30, 1932. [Per Mr. R. A. Vatcher.]

## METHODS OF LEARNING GEOMETRICAL THEOREMS.\*

BY A. W. SIDDONS.

FIRST of all I should like to say that I am not responsible for the subject of this paper. Some member of the Association proposed that a discussion should be held on this subject and the Committee that arranges the programme for this meeting asked me to read a paper.

This Association was founded a little over sixty years ago, mainly by schoolmasters, and its main object was to advance mathematical knowledge by improving the teaching of mathematics in schools, so that I am only too glad to welcome any discussion on actual school work, and I hope that what I have to say may provoke a discussion to which many of you will contribute.

After promising to read the paper I sat down to consider what ground it should cover. It seems to me that the subject covers nearly the whole ground of geometry teaching up to the School Certificate stage, so the subject is wide.

It will not be uninteresting to look at the past for a few minutes. I started teaching in the last century, and there was much teaching at that time that had made little advance since the date of the foundation of the A.I.G.T. But for a moment I will go back still earlier and tell you two stories of the middle of the last century.

Canon J. M. Wilson once told me that when he was a boy, in writing out Euclid, proofs had to be reproduced with meticulous accuracy—to write  $\angle CBA$  instead of  $\angle ABC$  would have been counted a mistake.

The late Master of Jesus College, Cambridge, used to tell a story about his friend Dicky Webster, afterwards Sir Richard Webster, and later Lord Alverstone. After taking his degree before going to the Bar, he was a coach in Cambridge for a short time. On one occasion a pupil who had to pass an examination in Euclid asked Webster to pick out twenty likely theorems, later the list was shortened down (if I remember rightly) to eight: after the paper the pupil rushed up to Webster's rooms and burst out, "I am through, I am through, I got six of the ones you chose and I got them right to a comma"; he added, "I am not sure that I put the right letters at the corners, but I suppose that does not matter".

I don't want to pretend that that was the method of learning Euclid when I started teaching; but still I found that many of my early pupils professed never to have done a rider. There was much memory work, and but little understanding in some cases.

Now to come to the conditions that obtain to-day—I am thinking particularly of secondary schools and public schools. Of course, much geometry is done in elementary schools.

In the first stages of geometry a boy or girl has to learn to draw and has to learn the language of geometry. In the next stage comes

\* A discussion at the Annual Meeting of the Mathematical Association, 6th January, 1933.

the acquisition of geometrical facts, or theorems if you like. The child should write out the enunciations of the theorems found and learn the enunciations by heart. But how are the theorems to be discovered, and how are children to be convinced of their truth? By measurement sometimes, by intuition more often, and sometimes by logical arguments, by logical proof.

Some people say that children have no logic in them; that I flatly deny. They have logic inside them, and it is the teacher's business to help them to develop and use that logical power.

Almost from the first there are theorems that they can prove logically. Vertical opposite angles—they see it, but they can appreciate the logical proof. Corresponding angles for parallels they can see by intuition; from that they can deduce the alternate angle property. The angle sum of a triangle—it is nice to gain first by experiment perhaps, but the logical proof is quite easy and will appeal.

All through the stage of acquiring a knowledge of facts or theorems some logical proof will occur, even in a first year at geometry in a secondary school or a preparatory school. Easy riders should be done, and the child should learn the shape of a logical proof and should write out the proofs of selected theorems and riders in proper form. The desire for logical proof should be created and should grow. To my mind, the child has not really acquired a new theorem until (i) he is convinced of its truth, (ii) he knows its enunciation, (iii) he can apply it to numerical cases, and (iv) he can use it, if need be, in a logical proof.

I spoke a few moments ago of the child writing out selected theorems in proper form. That raises a point on which there may be some discussion to-day. How far should we insist in these days on the strict formality of Data, To prove that . . . , Construction, Proof? I think we should keep them. They are actually a help, they make the work precise, and they are a training in English.

As soon as the child has won the properties of parallel lines and congruent triangles, the essential machinery of logical geometry is available. We have then to learn the properties of parallelograms, the mid-point theorems, areas and Pythagoras, the circle. Still our methods are measurement, intuition, and also logic; but more and more our pupils should see the logical connection between the theorems in a group; more and more they should be led to desire logical proof rather than mere experiment and intuition.

So far, the proofs of theorems have not been our main aim, but the child should already have

- (i) a list of theorems the truth of which he believes;
- (ii) the ability to write out the ultimate proof of a good many of these.

I make out that out of fifty theorems that will have to be learnt (that is, up to the end of the circle), thirty will already be known as results. Eighteen of these the child should have proved logically but will not necessarily remember the proof.

This stage should be reached two years before the child will take the School Certificate. That this is possible with the majority of children has been shown again and again ; but even if only one year is left, the child ought to be able to come up to the scratch in time for the School Certificate examination. Assuming that the child has two years, I would plunge at once into the learning of theorems. This work is really one of the easiest tasks that the teacher has, provided (and I grant that it is a big provision) that the previous work has been properly done. I should divide the theorems to be learned into three groups :

(i) Roughly, *Euclid*, Book I.

Angles at a point, parallels, congruent triangles, parallelograms, ruler and compass constructions (if I may include them), loci. (Inequalities may be taken in this year or left till the Certificate year.)

(ii) Areas and Pythagoras.

(The extensions and Apollonius might be left till the Certificate year.)

(iii) Circle group.

I do not think it matters much which group is taken first, but I would give a term to each, and in one term of that year I would do similarity. The last year before the Certificate would ideally be left for revision, and would really cover the same ground as the previous year, but harder examples could be done.

Now, I should like to consider these groups in turn, but time will not permit me to do that fully. Let us consider the first group briefly.

Happily most examining bodies follow the recommendation of the Board of Education and do not require the proofs of some of the fundamental theorems. Those theorems are the hardest to prove, and the proofs are often unlike anything else the boy will have to do, and perhaps the proofs are the least satisfactory. But it is very essential that the boy should realise that these theorems are part of the fundamental assumptions, part of the foundation on which he has to build.

The theorems, proofs of which are generally not required, are those about angles at a point, parallels, congruent triangles.

Of those that the boy must be able to prove, the angle-sum of a triangle presents no difficulty. The isosceles triangle theorems present the difficulty (i) of hypothetical construction ; and (ii), why bisect the angle rather than the base ?

The group of theorems I have mentioned so far, though they do not include many of which proofs are required, need a lot of attention. The boy should have some definite idea of the order in which they come.

(i) 2 line theorems ;

(ii) 2 parallel lines and a cutting line ;

(iii) triangle (3 line) theorems ; (a) the angle sum of a triangle, (b) congruence and isosceles triangles.

A lecture with questions about them at an interval of a fortnight and again at the end of the term.

*Inequalities* are a side issue and can be treated as a separate group at any time. The only difficulty is to remember which is the fundamental theorem.

The *parallelogram* group the boy will have already done as riders. The definition must be rubbed in hard, and if the boy has been properly taught in the earlier stage, he can write out the proof of any part (except one) of either theorem, and if he cannot there is little hope. Several of the cases must be written out so as to see that the boy knows the shape of a proof and is logical. Incidentally he must be told always to name the parallels when he is saying that a pair of alternate angles are equal.

Here let me digress a little on method.

I never teach geometry unless every boy has beside him a piece of scrap-paper.

Suppose the work in hand is a rider from the book or a theorem. I let every boy try to draw his own figure. This translation from the words of the book into a figure needs a lot of practice. The boy learns little from just watching me do it ; but, if he tries himself, he may learn a good deal from watching me. Incidentally, the boy's difficulty is more effectually dealt with because it is dealt with at once and the boy is saved from waste of time.

*N.B.*—I don't let the boy put his own letters in the figure ; I make him take the same letters that appear on the blackboard.

When I have drawn my figure, there may be questions the boys want to ask about the figure, and some boys will need to draw new figures.

With a theorem I follow the same procedure. Well ! we have got as far as the figure.

Each boy then tries to prove the rider or theorem, not writing it out but merely marking things in his figure. After a few moments I cross-question the class, "What have we got to do ?" I may get the answer, "All that we need do is prove those two triangles congruent".

"Each of you write down the three equalities that would do that."

After a minute or two, *viva voce* work is taken up, and I mark in my figure the things the boys tell me to mark. Any mistakes are dealt with at once, and the class should be able to write out the complete proof if called on to do so. Occasionally it is well to let them write out the proof at once, but I seldom do this.

More generally I discuss two or three allied theorems in this way, and then at the next or a subsequent lesson let them write out one of them. Generally I suggest that they should glance through the proof in the book after the lesson I have sketched above, and before they write it out. Sometimes there is an interval of a day, sometimes of a week or two weeks, between the lesson in which we discussed the theorem and the lesson in which it is written out.

If boys have been well grounded and logical power is well developed, the writing out of theorems depends mainly on remembering

where the theorem comes, what theorems precede it. Therefore the master's task is to dwell on the chain of theorems to drive home the perspective of the theorems as a whole or of a group. The odd two minutes at the end of a lesson can often be used to tickle their memories. "Draw the figure for the fundamental theorem in such-and-such a group."

A good lesson is to go through the whole lot of the theorems the child has to know and ask what is the fundamental theorem of the various groups.

"What is the fundamental theorem about tangents?"

"Give the enunciation of some other theorem about tangents."

"What is the fundamental theorem in the angles of a circle group?"

"What is the fundamental theorem about areas?"

"What is the fundamental theorem about similar figures?"

Once they know which is the fundamental theorem, the order of the rest of most groups of theorems does not matter much.

Then we have lessons on single groups of theorems.

Take the angle properties of a circle.

I have two methods which I use at intervals of two or three weeks.

(a) "What is the fundamental theorem of the angles in a circle group?"

Each boy has scrap-paper beside him and draws the figure. Mark in the figure the various facts that you will use: "Are you convinced that you could get full marks for it?"

I draw my figure on the board (one case only).

"What facts do we use?" Probably we go through the proof.

"How many of you remember that there are two cases? Full marks 8, 3 off if you only did one case. How many of you could get 8? How many 5?"

"How shall we describe the cases?"

"Now write down the enunciation of, or draw the figure for, a theorem that depends on this."

We discuss that, and then another, ultimately getting up to the alternate segment theorem. At subsequent lessons the class will write out one or two of these theorems.

(b) Two or three weeks later, I shall go over the whole group again, probably asking which is the last theorem of the group; on what theorems does it depend? How do we prove it? Then take one of those theorems and so work back to the fundamental.

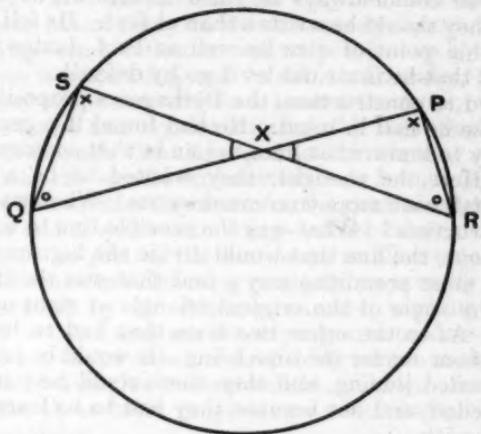
In this way the boy gets into his mind the order of the theorems in the group, and if he has that he should be able to write out any individual theorem.

But I ought to have said that there are two theorems in this group the proofs of which I should have ignored so far, namely, the converses of angles in the same segment and of the cyclic quadrilateral; probably leave them till the Certificate year. They present more serious difficulties and need slightly different treatment. First there is the enunciation of the first one. Each boy tries to state it and so

we learn that it is not easy to state, and eventually the enunciation is learnt by heart. Then the proof is discussed fully and we take the other theorem in the same way. I then tell the class to glance at the proof in the book and a few days later they try to write it out.

I might say a lot about other groups or individual theorems. For example, I might say that after doing Pythagoras, I let each boy try to draw the figure, but I do not give them the theorem to write out until every boy can draw the figure and feels pretty sure that he has the argument complete.

Here is one small point about similar triangles. When a boy has proved that two triangles are equiangular with one another in such a figure as here shown, I tell the class to write  $\frac{XP}{XR} = \frac{PR}{PR}$ , and



then write the corresponding sides underneath. Perhaps I ought not to say this as Professor Lodge is not here to express the opposite view, but I do not think it is wise to burden the child with considering whether to write  $XS$  or  $SX$ ; for me it is enough if the child writes down the side opposite the proper angle.

As I said, I might say much more, but I think that I might sum up what I do about learning theorems briefly as follows :

- (i) I aim at developing power and geometrical knowledge ;
- (ii) by frequent discussion of whole groups of theorems, I aim at giving the boy a general perspective view of the theorems in a group and so a knowledge of the order in which they come ;
- (iii) after that I trust to luck and do not, I hope, descend to cram.

#### DISCUSSION.

Mr. W. Hope-Jones (Eton) said that in the absence of Prof. Lodge he would endeavour to maintain the point of view he shared with him, though he was not accustomed to be able to state it as well as

Prof. Lodge himself. He referred to the naming of triangles. He believed in the importance of naming the triangles so as to make it clear what sides and corners of one corresponded to what sides and corners of the other. It might be a little bit of a burden to start with, but it saved boys, especially stupid boys, from the kind of mistake they were most likely to make later on. If, in addition to that, all the sides were called by single letters, the difficulty in the writing down of the equation disappeared, and when it came to writing down the names of the original triangles, a great deal of help was afforded by the use of colours. Of course, the marks of different kinds in the angles did help, but if colour were used to differentiate the angles it helped still more, and by that expedient, he thought, the difficulty would be found to disappear. As for the names of triangles, these could always be filled in afterwards when it was clearer how they should be written than at first. He felt that he had not stated this point of view as well as Prof. Lodge would have stated it, but that he must not let it go by default.

With regard to construction, the Pythagoras proposition was the particular case he had in mind. He had found it a great difficulty as a small boy to know what lines to join in that construction at the beginning. Here, he thought, they wanted to be a little more merciful and fall back more on common sense. What was the object of this construction? What was the sensible line to draw? Only one was obvious, the line that would divide the big square into two pieces in the most promising way; and that was the line that ran from the right angle of the original triangle at right angles to the hypotenuse. As to the other two lines that had to be joined, he would leave them out for the time being. It would be found later on what lines needed joining, and then these could be joined because they were needed, and not because they had to be learned out of a book to start with.

Miss E. F. Edwards (Fairfield High School) pointed out that children were helped to remember the construction for proving Pythagoras' theorem if they were shown that the second triangle was obtained by twisting the first through a right angle.

Mr. F. C. Boon (Dulwich College) said that a rather curious point occurred to him, having regard to Professor Lodge's insistence upon the naming of the triangles in the proper order. It was that this led to a very simple cramming device to which both Professor Lodge and Mr. Siddons would object. The speaker exemplified on the blackboard how one triangle might be taken, and its name written down, and underneath it the name of the corresponding triangle might be written according to the marks put for the angles. Thus if in the triangles  $ABC$  and  $PQR$ ,  $\angle A = \angle P$ ,  $\angle B = \angle Q$ , and  $\angle C = \angle R$ , one might write down the three sides of the first triangle, not choosing any particular order and underneath write the corresponding letters of the second:

$$\frac{BC}{QR}, \frac{CA}{RP}, \frac{AB}{PQ}.$$

One might then insert signs of equality between the fractions and obtain the statement of proportionality for the sides. The boy in that case need not understand the proposition at all.

Asked by a member of the audience whether that was a process they encouraged in algebra, Mr. Boon replied, "Not I".

Mr. W. C. Fletcher said that he would just like to emphasize a point which Mr. Siddons had made. In former days the first thing to learn was the propositions and their proofs—possibly some day one might be able to do riders, more probably not. Modern practice really begins with riders and takes the proofs of propositions more or less incidentally. What Mr. Siddons said was relevant and true, but perhaps he might have stressed more strongly the fact that proofs of propositions do need learning—though this is better done late than early. The difference is that they can then be constructed instead of being taken ready made and merely learnt. Very often the whole thing—enunciation as well as proof—can be constructed, and the process is interesting and valuable. Consider, for instance, the formulation of the converse to a known theorem; it is an easy step to draw a figure and say, for instance, "if these angles are equal a circle will go through these four points"; it is easy again to add letters and state the special enunciation in a formal fashion; it is harder to evolve a satisfactory form for the general enunciation, but it can be done and is well worth doing—a first-rate exercise in composition. As to the proofs, the best way, at all events for revision—and the whole matter of learning propositions is really revision—is that recommended by Mr. Siddons, namely, working backwards. Start where you like, and ask what propositions does this depend upon? Working backwards one inevitably comes in the end either to congruence of triangles or to parallels, or both. And this is the time to explore these basic propositions which have hitherto been accepted as intuitive.

Mr. M. P. Meshenberg (Tiffin School) had been rather disappointed at the absence of any guide in Mr. Siddons' paper to the manner of dealing with one or two groups of propositions that usually had a poor background, especially as one of them seemed to be steadily dropping out of favour, namely, the group referred to in the first paper read that morning, the semi-algebraical theorems, as he liked to call them. There seemed to be a general agreement that the case went by default, that these theorems were best treated as mere pictorial representations of algebraical facts, and did not seem to have any independent geometrical existence. But they arose out of practical necessities, perhaps as much as the time-honoured Pythagoras theorem. The prime object of all these theorems, including Pythagoras and its extensions, also Apollonius, as well as (possibly) the rectangle theorem of the circle, was the necessity for replacing distances measured from inaccessible points by distances measured from accessible points. That must have arisen quite early in constructional and surveying work, and the basic formula arose that distances measured from the ends of a line—that is, from two points which were inaccessible—to any other point might be

replaced by distances measured from the mid-point of that line. If the line were called  $AB$  and the middle point  $O$ , distances measured from any point to  $A$  and  $B$  could be replaced by distances measured from  $O$ . The advantage that he had found in that method of treatment was that it made a whole host of theorems—very unpleasant to teach separately, and much more unpleasant for the boys to learn—into a unified whole. The extension of Pythagoras, the Apollonius theorem, the rectangle property of the circle, and even of co-axal circles were obtained in a manner which replaced the distances measured from the ends of a line to a point by distances measured from the mid-point or from the centre of the circle.

He wished to mention one other point as a suggestion that occurred to him, namely, in connection with the opening of geometry by discussing the intersection of two lines, then, by parallels, of three lines, and thus getting to the triangles. It happened to contrast with his own method for many years of starting geometry by considering the neighbourhood of one point, its "morphology"—if he might use such an expression, borrowed from another science—by drawing rays from the point and discussing the angles; then of two points. That introduced parallels as the only device for translating the "morphology" of one point to the "morphology" of another. Then they went on to the geometry of three points, when they made the thrilling discovery that the three basic angles, those introduced by joining the three points, always had a fixed sum. It was a startling theorem, and extended itself naturally into the geometry of many points.

Mr. J. Katz (Selhurst Grammar School) said that Mr. Siddons had very rightly concentrated on how the proposition should be learned in the presence of the teacher. He rather wanted to raise the point of how the proposition should be learned in the absence of the teacher. Suppose that a boy had done the proposition with the class and the teacher. There must come a time when he would want to get at the proposition by himself. He would, of course, have his notes and text-book. This boy needed a certain amount of help. He would like to offer one or two psychological "tips", if he might call them so, as to how to set about revising the proposition. He would suggest, in the first place, that the boy ought not to draw the whole figure. If he understood Mr. Siddons aright, he very often tested the class rapidly by saying, "Who can draw the figure for this proposition?" He presumed that by that he meant the whole figure. It seemed to him that when the boy was revising the proposition by himself or preparing it for the morrow's lesson, he ought not to draw the whole figure, but ought to draw it piece by piece, putting in the various lines or points which he required as the proof developed. Otherwise one was tempting him to learn the whole proposition by heart.

In that connection he wanted to make a rather special point. Ought one not to abandon the tradition of having the so-called construction as a distinctive formal element in the proof? A theorem in this respect was quite different from a problem. In a problem one

had to invent the construction, but in the case of a theorem one was using the construction to assist the proof. If one was going to ask the boy to state the whole construction first, one was tempting him to learn the proposition rather unintelligently, which it was not desirable that he should do. Therefore he would make a case for omitting the construction as a distinct formal element in a proof. Let the construction appear in the proof at the point at which it was required. He would like to hear Mr. Siddons' opinion upon that.

His other point was that he thought they ought as far as possible to encourage the use of the letters of the Greek alphabet,  $\alpha$ ,  $\beta$ ,  $\gamma$ , and so on, for angles, instead of the angle  $ABC$ . It would save time in both writing and speaking. He did not know what the objections were to a very considerably enlarged use of Greek letters in place of the old-fashioned "angle  $ABC$ ".

Mr. Roebuck (Swanbourne House School) said that with regard to the old construction headings, he would like to disagree with the last speaker (Mr. Katz) on the ground that one of the difficulties, with the not too intelligent boy, was that he was very apt to include in his given facts just those lines which he presumably was drawing in order to help out his proof, and which he should think of much later. As soon as he was getting rather familiar with that proposition he did somewhat tend to put everything down as "given", and in the end succeeded in showing that he was unable to give anything of logical value at all. The speaker thought "Added lines" a better heading than "Construction", for again there were boys who found it rather difficult not to confuse the "construction" lines added to a diagram to help out a proof, with an accurate construction proper.

Mr. R. M. Wright (Winchester) said that it might be interesting to some among the audience if he drew a parallel with another kind of writing-out in which one had to get lucidity of expression above all things and also exactness of sequence, namely, the writing out of military orders. Military orders, according to the text-books, had to be written in the following sequence: information about the enemy, information about own troops, intention of the commander, and method of carrying it out. If these were transferred to the field under discussion, the first thing, the enemy, obviously corresponded exactly to the examiner; "own troops" was one's own construction to help one in defeating the examiner; the "intention" was quite clearly what was required to be proved, which was the objective, and the method of carrying it out was the proof. This analogy did show the necessity of having a clear and formal sequence, and if anything else was put into a similar framework: e.g. an essay, or a movement in a musical symphony—it would be found that one got a pattern which did appeal easily to the ordinary mind.

Mr. S. Lister (County School for Boys, Gravesend) desired to mention one or two things which had arisen as a matter of practical experience. In writing out theorems and riders, if a boy were asked to divide his page into four portions by lines down the middle of the page and across the page, he might put the figure in the top left-hand

space, the enunciation and construction in the top right-hand space, the proof in the bottom left-hand space under the figure, leaving the fourth space for the reasons. If this space were left empty it would be an obvious reminder that the reasons for the steps of the proof had not been given. Instead of having to write out the full enunciation of a reason, if one managed to abbreviate the reasons, it was a good thing for the boy himself. The speaker, by illustration on the blackboard, gave examples of his own abbreviations ; he used, for instance, the sign ||| for similar triangles. By the use of various abbreviations tremendous help was obtained in writing out, and the writing was speeded up, which was an advantage.

One other point was that few pupils realised that there were theorems whose converses were not true. He drew one figure on the blackboard—the angle at the centre of a circle and an angle at the circumference standing on the same arc. The converse of the theorem illustrated by that figure, as he demonstrated, was not true, but the pupil, in nine cases out of ten, would say it was, and he himself had received some ingenious proofs using the fact that the point he indicated was the centre of the circle, when this was not necessarily the case.

Mr. N. F. Sheppard (Lord Wandsworth Agricultural College) said that he believed that in German schools the ambiguous case was not stated as in this country. He demonstrated on the blackboard that there need be no ambiguity at all in certain cases in which there was not a right angle. The Germans stated that if two sides of a triangle and the angle opposite the greater of those sides were given, then there was no ambiguity. He could see no objection to our teaching the same fact. He would like to be enlightened upon that point.

Mr. Siddons, in replying on the discussion, remarked that his paper must have been a success, because it had kept people talking for the exact time scheduled for the discussion. Of course, many of the things which participants in the discussion had stated were things which he himself would have liked to have said had there been time. With regard to the question of colours, had he thought of it, he would have made an excursion to his classroom before coming to the meeting and have provided himself with coloured chalks. He begged them by all means to use coloured chalks. With regard to Mr. Hope-Jones's suggestion about the dividing up of the square of the Pythagoras theorem into two figures, he used both those, and he also appreciated Miss Edwards's suggestion.

As to the importance of learning a proposition, he felt that this was important, but that it came after the riders, and here he wished to reply to Miss Seymour, who had asked whether backward children should ever learn propositions. It depended on whether they could ever do riders. It was of no use their doing propositions until they could do riders. With a good teacher it would be longer before they got to propositions, but many teachers would cram them with propositions. For his own part he would not descend to teaching a boy to learn propositions until he could do riders. He met two years ago

one of his old boys, who said, " You don't remember me ? " He replied, " Yes, I do. You are the boy who most nearly defeated me in teaching geometry ". He was very nice about it and said, " I know I was pretty awful ".

With regard to data, and how formal they should be, for riders he did not mind being less formal, but for theorems one should be gradually developing formality. One had to grow more and more formal.

He had been interested in Mr. Fletcher's remarks, and he wished that Mr. Fletcher had read a paper instead of himself.

Mr. Katz had talked about learning without a teacher. What he had said was quite true. The boy had to look at the book, of course. He quite agreed that the boy should build up his figure as he went on. He always told the boys, " Never read a book on geometry without a piece of paper beside you, and work from your own figure, not the figure in the book ". He had been interested in what Mr. Wright said on the subject of military orders. It was an excellent parallel. Writing out the theorem was a question of abbreviating reasons, and so on. All that was training in English. For example, instead of saying, " This angle equals that angle because they are vertically opposite ", it read far nicer to say : " This angle equals the vertical opposite angle so and so ".

Mr. Katz had raised the point whether constructions should be put out formally first or should be stated in the course of proof. For his own part he liked putting out a thing formally first. In the case of a theorem the boy was familiar with the whole thing, but he would like to stick to the formal way of putting the thing out.

He had been interested to see the abbreviated references to the congruence theorems cropping up again which Mr. Lister had demonstrated on the blackboard. He believed they were originally due to a master at Osborne, Mr. Price, and he had always given him great honour on that account. He had never come to the conclusion that it should be given as a formal way of putting a reason, but he used it, and had advocated it being used in order that the boy might show that he did really understand which theorem he was using.

Mr. M. P. Meshenberg said that before the discussion closed, he did not think the Mathematical Association should allow it to go by default that " some theorems had no converses ", nor should they speak of " the converse " of a theorem, for a theorem could have many converses, as many as there were facts in the statement which could be interchanged with the fact required. There were one or two theorems in elementary geometry which had three distinct converses, all of them true.

On the motion of the President, a hearty vote of thanks was accorded to Mr. Siddons for his paper.

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207. Nulle courbe, ni nulle droite réelle ne peut passer entre deux lignes réelles qui se touchent ; ce ne sont-là que des jeux de l'entendement, des chimères idéales ; et la véritable géométrie est l'art de mesurer les choses existantes.—Voltaire, *L'homme aux quarante écus*. [Per Mr. J. B. Bretherton.]

## APPROXIMATING TO A SQUARE ROOT.

BY R. F. MUIRHEAD.

THE method explained by Dr. C. V. Boys \* has great merit. I for one fully sympathise with his preference for vulgarity in fractions. What here follows is meant to be to some extent supplementary to Dr. Boys' article.

If we denote by  $P_1/Q_1, P_2/Q_2, \dots$  the successive convergents to  $\sqrt{M}$  in Dr. Boys' series which are arithmetic means, and  $P_1=p, Q_1=q$ , the law connecting them is

$$P_{r+1} = P_r^2 + MQ_r^2, \quad Q_{r+1} = 2P_rQ_r.$$

In particular  $P_2/Q_2 = (p^2 + MQ^2)/2pq$ , which may be written

$$(pP_1 + MQQ_1)/(qP_1 + pQ_1).$$

This suggests another law of convergents :

$$P_{r+1}/Q_{r+1} = (pP_r + MQQ_r)/(qP_r + pQ_r).$$

The chief advantage of this is that we can get simple formulae for  $P_r, Q_r$ , namely

$$P_r = p^r + c_2p^{r-2}MQ^2 + c_4p^{r-4}M^2Q^4 + \dots + c_{2m}p^{r-2m}M^mQ^{2m} + \dots,$$

$$Q_r = c_1p^{r-1}q + c_2p^{r-3}MQ^3 + \dots + c_{2m+1}p^{r-2m-1}M^mQ^{2m+1} + \dots,$$

where  $c_m$  is the coefficient of  $x^m$  in the expansion of  $(1+x)^r$ . These formulae are easily verified by induction.

Thus  $P_4/Q_4 = (p^4 + 6p^2MQ^2 + M^2Q^4)/(4p^3q + 4pMQ^3)$ .

In the case of Dr. Boys' example, where  $M=10$ ,  $p=19$  and  $q=6$  this becomes

$(361^2 + 6 \cdot 361 \cdot 360 + 360^2)/4 \cdot 6 \cdot 19 \cdot (361 + 360) = 1039681/328776$ , and by division we get  $3.16227760169 \dots$ , so that the error does not affect the 11th decimal place.

To discuss further the nature of  $P_r/Q_r$  we observe that if

$$\alpha = (p - q\sqrt{M}) \quad \text{and} \quad \beta = (p + q\sqrt{M}),$$

then  $P_r + Q_r\sqrt{M} = \beta^r$  and  $P_r - Q_r\sqrt{M} = \alpha^r$ , and  $\alpha\beta = p^2 - q^2M$ , so that  $\alpha = (p^2 - q^2M)/\beta$ .

Hence  $P_r/Q_r\sqrt{M} = (\beta^r + \alpha^r)/(\beta^r - \alpha^r)$

$$= 1 + \frac{2\alpha^r}{\beta^r - \alpha^r}.$$

Thus  $P_r/Q_r = \sqrt{M} + \frac{2\alpha^r\sqrt{M}}{\beta^r - \alpha^r}$ .

Thus the error in excess by taking  $P_r/Q_r$  for  $\sqrt{M}$  is  $2\alpha^r\sqrt{M}/(\beta^r - \alpha^r)$ . Now  $\beta^r$  tends to be very large, and therefore  $\alpha^r$  to be very small, so that a close approximation to the error is given by  $2\alpha^r\sqrt{M}/\beta^r$ .

\* *Math. Gazette*, XVI (May, 1932), p. 111.

Further,  $\sqrt{M}$  approximates to  $P_r/Q_r$ , and

$$a = (p^2 - q^2 M) / (p + q\sqrt{M})$$

which approximates to  $(p^2 - q^2 M) / 2p$ . Thus a fairly close approximation to the error (which is always in excess) is

$$2p(p^2 - q^2 M) / (2p)^2.$$

So going back to the numerical example already considered, the error in excess by taking  $P_4/Q_4$  is about  $2 \cdot 19 / (6 \cdot 38^2)$ , which is less than  $13 \times 10^{-11}$ . If a closer approximation is required, we can use, say,  $P_8/Q_8$ , where

$$P_8 = p^8 + 280 \cdot 361^3 \cdot 360 + 7000 \cdot 361^2 \cdot 360^2 + 28000 \cdot 361 \cdot 360^3 + 10000 \cdot 360^4,$$

$$Q_8 = 8 \cdot 19 \cdot 6 \cdot (361^3 + 70 \cdot 361^2 \cdot 360 + 700 \cdot 361 \cdot 360^2 + 1000 \cdot 360^3),$$

and the error would be approximately  $1/(6 \cdot 38^8)$  which would not affect the 24th decimal place.

An alternative method of getting a closer approximation would be to iterate the process of deriving  $P_r/Q_r$  from  $p/q$ . This would give

$$(P_4^4 + 6P_4^2MQ_4^2 + M^2Q_4^4) / 2P_4Q_4(P_4^2 + MQ_4^2),$$

or, in the case of  $\sqrt{10}$ ,

$$\frac{1039681^4 + 60 \cdot 1039681^2 \cdot 328776^2 + 100 \cdot 328776^4}{2 \cdot 1039681 \cdot 328776 \cdot (1039681^2 + 10 \cdot 328776^2)}$$

with an error in excess of about  $10^{-55}/8$ , which would not affect the 55th decimal place.

The convergence is most rapid when  $p$  and  $q$  are chosen (as in the above case) so that  $p^2 - Mq^2 = 1$ . Such values of  $p$  and  $q$  can always be found, when  $M$  is an integer, by the aid of continued fractions. We note that in this process, whatever value we choose for  $r$ , only one operation of "long division" is required, and that the closeness of the approximation attained is easily estimated. Dr. Boys' method is equivalent to a continued iteration of the step from  $P_1/Q_1$  to  $P_2/Q_2$ , and involves one "long division" for each step.

Since writing the preceding, it has occurred to me to seek for formulae giving the  $r$ th approximation and its error, in Dr. Boys' own method. Taking  $a = p - q\sqrt{M}$ ,  $\beta = p + q\sqrt{M}$ , we have

$$p = \frac{1}{2}(\beta + a), \quad q\sqrt{M} = \frac{1}{2}(\beta - a),$$

and the first approximation is  $p/q$  with error  $p/q - \sqrt{M}$  or  $a/q$ . The second is  $(p^2 + q^2 M) / 2pq$ .

Thus if we denote the numerator and the denominator of the  $r$ th approximation by  $P_r$  and  $Q_r$  (where these have now significations different from what they have in the previous part of this article), we have

$$P_2 = p^2 + q^2 M = \frac{1}{2}(a^2 + \beta^2); \quad Q_2 = (\beta^2 - a^2) / 2\sqrt{M},$$

$$P_2 = P_2^2 + Q_2^2 M = \{\frac{1}{2}(\beta^2 + a^2)\}^2 + \frac{1}{2}(\beta^2 - a^2) = \frac{1}{2}(\beta^4 + a^4),$$

$$Q_2 = 2P_2Q_2 = 2(\beta^2 + a^2)(\beta^2 - a^2) / 4\sqrt{M} = (\beta^4 - a^4) / 2\sqrt{M};$$

and, generally,

$$P_r = \frac{1}{2}(\beta^n + a^n), \quad Q_r = (\beta^n - a^n)/2\sqrt{M}, \quad \text{where } n = 2^{r-1}.$$

Thus

$$\frac{P_r}{Q_r} = \frac{\beta^n + a^n}{\beta^n - a^n} \sqrt{M} = \sqrt{M} + \frac{2a^n \sqrt{M}}{\beta^n - a^n}.$$

Thus the error in excess in taking  $P_r/Q_r$  for  $\sqrt{M}$  is  $2a^n \sqrt{M}/(\beta^n - a^n)$  and since  $a = (p^2 - q^2 M)/\beta$ , and  $\sqrt{M} = p/q$ ,  $\beta = 2p$  and  $(a/\beta)^n \rightarrow 0$ , a good approximation to the error is

$$\frac{2p(p^2 - q^2 M)^n}{q(2p)^{2n}}.$$

Also with the same meaning as before for  $c_m$ , we have

$$P_r = p^n + c_2 p^{n-2} M q^2 + c_4 p^{n-4} M^2 q^4 + \dots + c_{2m} p^{n-2m} M^m q^{2m} + \dots,$$

$$Q_r = c_1 p^{n-1} q + c_3 p^{n-3} M q^3 + \dots + c_{2m+1} p^{n-2m-1} M^m q^{2m+1} + \dots,$$

where, as before,  $n = 2^{r-1}$ .

R. F. M.

**908. DECEIVED.** Professor G. N. Watson, of Birmingham University, played a confidence trick on a hall crowded with people this evening. They had come to hear what he advertised as "The marquis and the land agent—a tale of the eighteenth century", surely the strangest title ever chosen for a presidential address to the Mathematical Society. Hoping to hear a good bedtime story for mathematicians, I went to the meeting and found it very full.

Blandly Professor Watson opened by saying he had chosen a "seductive title which does not have a very close connection with the actual subject". He raised our hopes again by promising he would keep off a subject on which he has spent the last two years, but which apparently is so abstruse that to talk about it even to mathematicians would on his own admission have emptied the hall.

Instead he gave an "elementary" account of work on arcs of ellipses and other curves which led to the theory of elliptic functions and doubly periodic functions generally.

As soon as Professor Watson put the lights out to show a slide I shot off at a tangent.—*Sheffield Telegraph*, 6th January, 1933. The next paragraph is headed "Zebra Frocks"! [Per G. N. W.]

**909.** "I'm going to teach Miriam algebra", he said. "Well", replied Mrs. Morel, "I hope she'll get fat on it".—D. H. Lawrence, *Sons and Lovers*, ch. 7. [Per Mr. P. J. Harris.]

**910.** I have often thought that if men would put into the search for Truth one hundredth part of the energy they put into the holding of their delusions, we'd have an enlightened world to-morrow. . . . I remember having the most terrible trouble with a lunatic a few years ago! . . . His fixed idea was that all life revolved round the basic principles of the straight line and the curve; and that until mankind—which mistakenly called them God and the Devil respectively—had returned to the worship of the straight line and banished the curve from everything, we'd all go hurtling to hell-fire! . . . He also held that Woman was more the centre of evil than Man, because, as long as her body retained its present form, curves would persist in Nature though banned in architecture. You can see how the coming of evil into the world with Woman can be worked out upon that theory!—Temple Lane, *The Bands of Orion*, bk. ii. ch. 4. [Per Mr. P. J. Harris.]

## PARAMETRIC EQUATIONS IN ELEMENTARY ANALYTICAL GEOMETRY.\*

By A. ROBSON.

MATHEMATICAL students are usually familiar with the parametric coordinates  $(at^2, 2at)$  of a point on a parabola, and with the eccentric-angle coordinates of a point on an ellipse, and with similar coordinates for a hyperbola. Such coordinates are probably welcomed by the student just because they reduce by one the number of letters and the number of equations with which he has to deal.

If  $at^2$  is substituted for  $x_1$ , and  $2at$  for  $y_1$ , but everything else is done in just the same way as before, parametric methods are not really being used. Parametric equations are something more than convenient coordinates.

I shall not attempt to give any general account of their use, but shall illustrate it by special cases such as that of the parabola. Two or three general results which will be mentioned are only intended as reminders for the teacher, not as proposed additions to the curriculum.

The very form of a typical question is in one way misleading :

"Find the equation of the tangent at the point  $(at_1^2, 2at_1)$  to the parabola  $y^2=4ax$ ",

translated into the language of parameters, becomes

"Find the equation of the tangent at the point  $t_1$  to the parabola  $x=at^2, y=2at$ ",

or better still, "... to the parabola  $x : y : a = t^2 : 2t : 1$ ".

Everybody knows that if  $t$  is eliminated between the parametric equations, the ordinary equation is obtained. What is apt to be forgotten is that  $t$  ought *not* to be eliminated.

Perhaps the process most typical of parametric methods is that of finding where the straight line

$$lx + my + na = 0, \dots \quad (1)$$

meets the curve

$$x : y : a = f(t) : g(t) : h(t).$$

The points are given by

$$lf(t) + mg(t) + nh(t) = 0, \dots \quad (2)$$

which is an equation for their parameters.

For example, to find the equation of the tangent at the point  $t_1$  to the parabola  $x : y : a = t^2 : 2t : 1$ , we have only to write down an equation which reduces to  $t^2 - 2t_1t + t_1^2 = 0$  for those values of  $x, y$ . It is  $x - t_1y + at_1^2 = 0$ .

Again to find where the tangent at the point  $t_1$  to the semi-cubical

\* A paper read at the Annual Meeting of the Mathematical Association, 6th January, 1933.

parabola  $x : y : a = t^2 : t^3 : 1$  meets the curve again, we use the equation  $lt^2 + mt^3 + n = 0$ , which has roots  $t_1, t_2, t_3$  such that

$$t_2t_3 + t_3t_1 + t_1t_2 = 0 ;$$

if  $t_3 \rightarrow t_1$ , this equation shows that  $t_3 \rightarrow -1t_1$ .

More generally, if  $f(t)$ ,  $g(t)$ ,  $h(t)$  are polynomials and  $t^m$  is the term of highest degree therein, equation (2) is of degree  $m$  in  $t$  and shows that the curve meets a straight line in  $m$  points. It is a curve of order  $m$ .

The condition of equal roots of (2) is found from

Equations (2) and (3) give

$$l:m:n = \begin{vmatrix} g & h \\ g' & h' \end{vmatrix} : \begin{vmatrix} h & f \\ h' & f' \end{vmatrix} : \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

which are parametric envelope equations of the curve.

Equations (1), (2), (3) give by elimination of  $l$ ,  $m$ ,  $n$

$$\begin{vmatrix} x & y & a \\ f(t) & g(t) & h(t) \\ f'(t) & g'(t) & h'(t) \end{vmatrix} = 0,$$

which is the equation of the tangent. This applies even when the functions  $f$ ,  $g$ ,  $h$  are not algebraic.

When  $f(t)$ ,  $g(t)$ ,  $h(t)$  are polynomials the curve is called rational, and it is interesting to enquire what curves are rational. Although this may be outside the scope of elementary analytical geometry, it need not be outside the mind of the teacher of that subject.

In the first place, every *conic* is rational. For example,

for  $y^2 = 4ax$ ,  $y/2a = 2x/y$ ,  $= t$ ; thus  $y = 2at$ ,  $x = at^2$

And an easy extension of this process leads to the parametric equations of any conic. For the equation of a conic may be written  $\beta^2 = \alpha\gamma$ , where  $\alpha = 0$ ,  $\gamma = 0$ ,  $\beta = 0$  are two tangents and their chord of contact; putting  $\beta/\gamma = \alpha/\beta = t$ , we get

$$t^2 : t : 1 = a : \beta : \gamma \equiv a_1 x + b_1 y + c_1 : a_2 x + b_2 y + c_2 : a_3 x + b_3 y + c_3$$

whence

$$x:y:1 = A_1 t^2 + A_2 t + A_3 : B_1 t^2 + B_2 t + B_3 : C_1 t^2 + C_2 t + C_3; \dots \dots \dots (4)$$

thus every conic has rational parametric equations

The usual equations  $x=a \cos \phi$ ,  $y=b \sin \phi$  of an ellipse are special cases of (4), since they can be written

$$x/a : y/b : 1 = 1 - t^2 : 2t : 1 + t^2.$$

Similarly for the hyperbola.

Another special case is the parabola  $x=at^2$ ,  $y=b(1-t)^2$ , which used to be written in the form

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1.$$



Thus the usual details about a conic can be found by simple methods from parametric equations.

If  $\phi_m\{x, y, 1\}=0$  is the general algebraic curve of order  $m$ , the conic (4) meets it where

$$\phi_m\{A_1t^2 + A_2t + A_3, B_1t^2 + B_2t + B_3, C_1t^2 + C_2t + C_3\}=0,$$

and this is an equation of degree  $2m$  in  $t$ . For example a conic meets a quartic in eight points.

An alternative method of reaching parametric equations of a conic is to find where a line  $y=tx$  through an origin  $O$  on the conic meets the curve again. The substitution of  $tx$  for  $y$  in the equation of the conic gives a quadratic for  $x$  of which one root is zero; hence the other root is rational. Thus the  $x$  and  $y$  of a point on the conic are found rationally in terms of  $t$ . Moving the origin to an arbitrary point only adds constants to the values of  $x$  and  $y$ . This method may be applied to a cubic curve with a double point or cusp.

Non-singular cubics do not have rational parametric equations; for if such equations are given, a double point or cusp can be found, as in the following example :

The cubic

$$x : y : 1 = t^3 + 1 : t^3 - 1 : t + 1$$

meets  $lx + y + n = 0$ , where

$$l(t^2 + 1) + t^3 - 1 + n(t + 1) = 0,$$

i.e. in points whose parameters  $t_1, t_2, t_3$  satisfy

$$t_1 + t_2 + t_3 = -l, \quad t_2t_3 + t_3t_1 + t_1t_2 = n, \quad t_1t_2t_3 = 1 - l - n;$$

hence  $t_1t_2t_3 + t_2t_3 + t_3t_1 + t_1t_2 = t_1 + t_2 + t_3 + 1, \dots \dots \dots (9)$

is the condition of collinearity for three points on the cubic. If

$$t_1t_2 + t_2 + t_1 = 1 \quad \text{and} \quad t_1t_2 = t_1 + t_2 + 1, \dots \dots \dots (10)$$

then (9) is true for all values of  $t_3$ . This means that  $t_1, t_2$  are the parameters of a double point on the curve. The solution of (10) gives  $i$  and  $-i$  as the values of  $t_1, t_2$ ; thus  $(0, -1)$  is an (isolated) double point.

In general, a curve has rational parametric equations if it possesses as many double points and/or cusps as it can, having regard to its order. A non-degenerate curve of order  $m$  cannot have more than  $\frac{1}{2}(m-1)(m-2)$  double points, and if it has as many as that, it will have rational parametric equations.

For example, a cubic cannot have two double points, because the line joining them would cut the curve four times, twice at each double point.

And a quartic cannot have more than three double points, because if it had four the conic through these points and another point of the curve would meet the quartic nine times.

If a quartic has double points  $A, B, C$ , and  $D$  is another (fixed) point on it, a system of conics,  $s=ts'$ , passes through these four points. But  $s=ts'$  meets the quartic twice each at  $A, B, C$  and once

at  $D$ ; hence it meets it at only one other point; the coordinates of this other point can be found as rational functions of  $t$  by solving the equations of the conic and quartic.

As an actual example, we may take

$$x^2y^2 = x^2 - y^2,$$

C

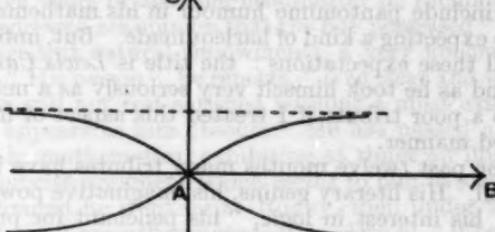


FIG. 2.

which has asymptotes  $y = \pm 1$ .

Near the origin  $x^2 \approx y^2$  and a second approximation is

$$\pm y = x - \frac{1}{2}x^3.$$

There are double points at infinity on the axes. If the point  $D$  is taken to tend to  $A$  along one branch of the curve,  $s$  and  $s'$  can be  $xy$  and  $(x+y)(z)$ ; thus we solve

$$x^2y^2 = (x^2 - y^2) \quad \text{and} \quad xy = t(x+y),$$

which gives

$$x : y : 2 = t + t^3 : t - t^3 : 1 - t^4.$$

[These are actually found more simply by putting  $x = \tan \phi$ ,  $y = \sin \phi$ .]

The more general results which have been kept in the background are evidently outside the scope of elementary teaching. But I have tried to show that elementary examples can be chosen which give some insight into the later work. And parametric methods are often appropriate in elementary analytical geometry even when that subject is limited to conics as, I fear, it usually is.

A. R.

911. One day he turned to a young, new master, as they were going into school, and said: "Can you turn this into Greek iambics?—'To find four points such that the line joining any two of them is at right angles to the line joining the other two'". At the same time he sent off the problem for solution to a late member of the Sixth who had gone up to Cambridge with a scholarship. The young master was ready next morning with his reply; the Cambridge scholar sent the solution by return of post—two people inspired at one blow. The solution was: the three points of an equilateral triangle, the fourth point being the centre of the triangle itself.—James M. Wilson: *An Autobiography* (1932), p. 172. [Per Mr. F. P. White. See also W. J. Greenstreet's review of Sir T. L. Heath's *Euclid in Greek*, Gazette, x. 349.]

## LEWIS CARROLL—MATHEMATICIAN.\*

BY D. B. EPERSON.

As this is the last paper of our meeting, you may be expecting me to provide a bright and witty epilogue to conclude the proceedings. Further, since Lewis Carroll is recognised as a humorist, who did not hesitate to include pantomime humour in his mathematical essays, you may be expecting a kind of harlequinade. But, unfortunately, I cannot fulfil these expectations: the title is *Lewis Carroll—mathematician*, and as he took himself very seriously as a mathematician it would be a poor tribute if I treated this aspect of his work in a light-hearted manner.

During the past twelve months many tributes have been paid to Lewis Carroll. His literary genius, his imaginative powers, his love of children, his interest in logic, "his penchant for puns, puzzles, parodies and palaver",† all these have been the subject of newspaper and magazine articles, as well as of the more ambitious essays and books in praise of one who always avoided publicity, and was almost ashamed of being recognised as the author of *Alice in Wonderland*. But little has been heard of the Rev. C. L. Dodgson, mathematical lecturer of Christ Church, Oxford, the author of mathematical pamphlets and books, which he was not ashamed to publish without a pseudonym.

A writer in the *National Review* at the time of his death, after eulogising the talents of Lewis Carroll, prophesied that "future generations will not waste a single thought upon the Rev. C. L. Dodgson".

This gloomy prophet was a false prophet: for to-day, if the correspondence columns of our leading newspapers can be relied upon as evidence of public interest, our chief aim is to pierce behind the veil of pseudonymity and discover the personality of the Rev. C. L. Dodgson. Some writers, troubled by the apparent differences between Lewis Carroll and Charles Dodgson have had recourse to a theory of dual personality; such fantastic explanations are due to the fact that their inventors, knowing little of the private life of Dodgson beyond the fact that he was a mathematician and a parson, immediately jump to the unwarrantable conclusion that he must have been dull and priggish. The problem of explaining how such a man could have been the creator of "Alice" is really a non-existent one.

One has only to read some of Dodgson's logical and mathematical books, such as *The Game of Logic* and *Euclid and his Modern Rivals*, to perceive that the mind that created the Mock Turtle and the Mad Hatter is still at work. The truth is that Dodgson's "personality was a completely integrated one",‡ and it is no exaggeration to say

\* A paper read at the Annual Meeting of the Mathematical Association, 6th January, 1933.

† *Lewis Carroll*, by W. de la Mare.

‡ R. B. Braithwaite, *Math. Gazette*, No. 219.

that "no one but a mathematician could have written *Alice in Wonderland*".\*

There is no doubt that Dodgson regarded himself primarily as a mathematician, and that he hoped that his mathematical books would prove to be of permanent value and interest. From his earliest years he had shown marked mathematical ability. When he was twelve years of age, the headmaster of Richmond School, Yorkshire, expressed his opinion that the boy possessed "along with other and excellent natural endowments, a very uncommon share of genius". "His reason", he reports, "is so clear and so jealous of error, that he will not rest satisfied without a most exact solution of whatever appears to him obscure. He has passed an excellent examination in mathematics, exhibiting at times an illustration of that love of precise argument which seems to him natural."

From Richmond he went on to Rugby, then under the aegis of Dr. Tait, who had recently succeeded Dr. Arnold. When Dodgson left that famous school, he wrote in his diary that he did not look back upon his time at a Public School with pleasure, and that "nothing on earth would induce him to go through his three years again". He confesses, however, that the time had not altogether been wasted: "I made, I suppose, some progress in learning of various sorts, but none of it was done 'con amore'".

When he came into residence at Christ Church his troubles appear to have ended. From January 1851 till the day of his death—a period of forty-seven years—he remained a resident member of the House. As an undergraduate he took 1st class Honours in Mathematical Moderations, a 2nd in Classical Moderations, a 3rd in "Greats", and a 1st in Mathematical Finals, where his name appeared at the head of the list.

After taking his degree, he first occupied the humble post of Sub-Librarian, but was soon appointed to a Mathematical Lecturehip, which he retained for thirty-five years. Later he was elected a student of Christ Church and took Holy Orders in 1861. The rest of his life was spent in the calm and cloistered seclusion of Tom Quad: even when he unintentionally became world famous as "Lewis Carroll", Charles Dodgson remained in obscurity quietly continuing his work as lecturer and tutor, and entertaining his personal friends, some of them eminent Victorians—Tennyson, Rossetti, Ellen Terry, Holman Hunt, and others, like Alice Liddell, children who were his constant companions and admirers.

The first book published by Dodgson was in 1860, *A Syllabus of Plane Algebraic Geometry*. The title is self-explanatory: the book is an attempt to arrange in logical sequence the processes of analytical geometry. Evidently he felt that the usual methods of teaching the subject lacked the logical cohesion of Euclid's pure geometry, and his passion for strict accuracy, shown even whilst at school, led him to attempt to give definitions to the concepts of algebraic geometry and to outline the axioms on which the subject is based. In this same year a 6d. pamphlet appeared, entitled *Notes on Euclid*:

\* E. E. Kellett, in a review of Langford Reed's *Life of Lewis Carroll*.

in this he even out-Euclids Euclid by defining the terms "problem" and "theorem": "a problem is something to be done", whilst "a theorem is something to be believed, for which proof is given". I think it is doubtful if there is really much to be gained by definitions of this sort, but it is interesting to note Dodgson's love of logic, the study of which developed into one of his chief mental pursuits. The process of defining one's terms can be continued *ad infinitum et ad nauseam*. Why draw the line at defining "problem" and "theorem"? Why not also define "definition"?

Two more pamphlets appeared in the next year: *Notes on Algebra*, and *Formulae of Plane Trigonometry*. The first of these is elementary, but its very existence is a testimony to the author's scrupulous attention to the needs of his students who were beginners in algebra. The second book is far more original. It is printed with symbols instead of words to express the trigonometrical ratios. To quote from the preface: "Symbols are substituted for the cumbrous expressions, sin, tan, cosec, etc., each requiring two strokes of the pen, one the same in each,  $\wedge$ ." The symbols are ingenious, but simple:

sin    cos    tan

cosec   
 sec   
 cot 

The only possible criticism is that "cosec" and "sec" are too much alike, and a possible improvement would be to invert the sine and cosine symbols  $\text{U}$   $\text{J}$  for the cosec and sec respectively. Dodgson apparently was doubtful whether this proposed reform would prove acceptable to the mathematical world and asked for opinions and suggestions. It is not recorded whether any favourable replies were received, but as the symbolism has not been generally adopted, we must conclude that the mathematicians of 1861 failed to see its advantages. Another proposed reform was the use of the word "Goniometry", which is sufficiently ugly to condemn its use at once. He defines "Goniometry Proper" as "the measurement of angles by angular units", "Goniometry by Ratios" as "the indication (not measurement) of angles by what are called the 'Goniometrical Ratios'", whilst "Trigonometry" is concerned with "the properties of Rectilineal Figures". Dodgson had a flair for inventing new words, two of which have passed into the English language, "galumphing" and "chortle", but "goniometry" has fortunately passed into oblivion together with the "slythy toves" and "mome raths" of *Jabberwocky*.

An Elementary Treatise on Determinants (with applications to Simultaneous Linear Equations and Algebraic Geometry) appeared in 1867. It exhibits those same qualities of originality and logical consistency as his former books. It is a treatise for the specialist: apart from this, the book's sole claim to distinction is that it features in the apocryphal anecdote about Queen Victoria. After reading *Alice in Wonderland*, it is said that the Queen sent to the publishers

for any other books by Lewis Carroll and was surprised to receive this *Treatise on Determinants*. Doubtless she was not amused.

Previous to the publication in 1879 of *Euclid and his Modern Rivals*, Dodgson had written several smaller works dealing with Euclid's Books I to VI, including two books concerned with alternative algebraic methods of proving the ratio theorems of Book V, and an edition of Books I and II, containing such small alterations and additions as he thought sufficient to render them satisfactory for modern use. Other mathematicians, however, were of the opinion that more radical changes were necessary: amongst them was to be found Canon J. M. Wilson, who also published a Geometry book covering the same ground. The following quotation from Canon Wilson's *Autobiography* is interesting in this connection: "I drafted a syllabus of Geometry corresponding to the first two books of Euclid, and gave reasons in the preface, and then the war began. It became quite plain that we must have an "Association of Mathematical Teachers" or otherwise there would be confusion. This I set on foot: we first met at my house in Rugby. In 1871 we became the Mathematical Association".

In the fierce controversy that led to the founding of our Association, Dodgson emerged as the champion of Euclid, and proved himself a formidable antagonist to the Modern Rivals. His book, solemnly dedicated to the memory of Euclid, was the most widely read of all his mathematical publications, probably because of its inimitable style. In Dodgson's own words, "It is presented in a dramatic form, partly because it seemed a better way of exhibiting in alternation the arguments on two sides of the question, partly that I might feel myself at liberty to treat it in a rather lighter style than would have suited an essay". The *Dramatis Personae* include Minos, a perplexed examiner in Geometry, the Ghost of Euclid, and Herr Niemand, "a German Professor who has read all books and is ready to defend any thesis, true or untrue": to him is committed the task of arguing the case for the Modern Rivals—Legendre, Cooley, Wilson, and at least a dozen others. The "Association for the Improvement of Geometrical Teaching" is parodied mercilessly. In Act III, the stage directions read: "Enter a phantasmic procession grouped about a banner on which is emblazoned in letters of gold the title, 'Association for the Improvement of Things in General'. Foremost in the line marches Nero, carrying his unfinished scheme for lighting and warming Rome, whilst amongst the crowd which follows him may be noticed Guy Fawkes, President of the Association for raising the position of Members of Parliament". It is difficult to summarise the contents of this book: its main objects are (1) to defend Euclid against the faults alleged by the reformers, faults of style, artificiality, unsuggestiveness, want of simplicity, etc.; (2) to show the lack of logical consistency in the modern Geometries; (3) to demonstrate that Euclid's Axioms and Definitions could not be improved; (4) to insist that the order and numbering of theorems in Euclid's system should be preserved, alternative proofs and additional theorems being inserted where necessary. One of the

chief subjects discussed is the treatment of the Theory of Parallels and the substitution of Playfair's axiom for Euclid's Axiom 12, and other alternative axioms such as Wilson's, based upon the conception of "lines in the same direction". In every case Professor Niemand, the defender of the Modernists, is worsted, usually through getting tied up in the logical knots skilfully prepared by Minoe. Wilson's *Elementary Geometry* is shown to contain seven unaxiomatic axioms, six instances of *Petitio Principii*, and many other specimens of fallacious reasoning, and so its claim to be a model of logical precision intended to supersede Euclid is dismissed as "simply monstrous".

Over fifty years have elapsed since Dodgson wrote this witty satire, but Euclid is no longer found in our schools, and the successors of the Modern Rivals seem to have triumphed. Was Dodgson's book, therefore, a failure—inasmuch as he was defending one of the lost causes which have found their home in Oxford? Such a verdict would be unjust. Emulating the judicious Dodo, it would be fairer to adjudicate that "Everybody has won, and all shall have prizes". Dodgson certainly showed the superiority of Euclid's Geometry as a logical treatise, whilst the Modern Rivals have succeeded in providing us with less formal and less formidable text-books which are better suited to the interests and abilities of the adolescent mind.

On 13th October, 1881, Dodgson wrote in his diary: "I have just taken an important step in life, by sending to the Dean a proposal to resign the Mathematical Lectureship at the end of this year. I shall now have my whole time at my own disposal and, if God gives me continued health and strength, may hope, before my powers fail, to do some worthy work in writing—partly in the cause of mathematical education, partly in the cause of innocent recreation for children, and partly I hope (though utterly unworthy of being allowed to take up such work) in the cause of religious thought".

These fair hopes were in vain; the two slim volumes entitled *Curiosa Mathematica* and an edition of *Euclid*, Books I and II (which went through seven editions), were his sole contributions to mathematical thought during the remainder of his busy life. Volume I of *Curiosa Mathematica* was sub-titled, *A New Theory of Parallels*, and is an ingenious attempt to avoid using Euclid's 12th axiom or Playfair's axiom by using an entirely original axiom of his own invention. The introduction to the first edition is interesting, as in it are discussed other matters besides parallel axioms. It records his controversial correspondence with two "Circle Squarers". "The first of these two misguided visionaries", says Dodgson, "filled me with a great ambition to do a feat I had never heard of as accomplished by man, namely to convince a circle squarer of his error! The value my friend selected for  $\pi$  was 3.2: the enormous error tempted me with the idea that it could be easily demonstrated to be an error. More than a score of letters were interchanged before I became sadly convinced that I had no chance."

After discussing what appear to him to be the shortcomings of

Euclid's Axiom, he proceeds : " And now to come to the end of this overlong Preface, am I not right in thinking that on mere inspection of the diagram



any sane intellect will be ready to grant that 'in any circle, the inscribed hexagon is greater than any one of the segments that lie outside' "? Using this axiom as his basis, and five other axioms about varying magnitudes, Dodgson is able to prove a succession of fifteen propositions, which lead to a proof that the three angles of every triangle are together equal to two right angles. From this he can easily prove the Euclidean Proposition that follows from his 12th axiom—that a pair of lines which are equally inclined to a certain transversal are so to any transversal.

In the third edition of the book he deals with various objections which had been made to his axiom and methods of argument, and also makes an important change in the axiom itself. The hexagon is replaced by a square, and we are asked to take it as axiomatic that "in any circle the inscribed tetragon is greater than any one of the segments which lie outside it".

An appendix bears the startling title, "Is Euclid's Axiom True?" His answer is, "It is true for finite magnitudes, the sense in which, as I believe, Euclid meant it to be taken. It is not universally true". Euclid's Axiom states that "if two lines are unequally inclined to a transversal they will meet when produced indefinitely". What happens, then, when the angle between the lines is an infinitesimal angle of the first order, or, worse still, of the second order? Do they meet? If so, where?

Dodgson is in deep waters in discussing this, and naturally assumes that Euclid's Geometry is the geometry of the space of the physical universe. He writes : "Will the gentle reader be so kind as to join me in contemplating the Infinite Space which surrounds our tiny planet. We believe it is Infinite. And that not because we profess to have grasped the idea of Infinity, but because the contrary hypothesis contradicts reason, and what contradicts reason we feel ourselves authorised to deny. Both conceptions—that Space has a limit, and that it has none—are *beyond* reason : but the former is also *against* reason, for we may fairly say : 'When we have reached the Limit, what then? What do we come to?'"

It would be unfair to blame Dodgson for not being acquainted with the yet unborn theories of Einstein, and he can hardly be

reproved for not visualising "a finite but unbounded universe". What his logical mind would have made of these modern concepts of space-time is difficult to imagine. Even the topsy-turvy world which Alice found through the Looking Glass is commonplace compared with a universe expanding like an inflated balloon of prodigious dimensions.

The second volume of *Curiosa Mathematica* bore the sub-title of "Pillow Problems thought out during sleepless nights". In the second edition, "sleepless nights" was changed to "wakeful hours" in order to allay the anxiety of his many friends who wrote to express their sympathy, believing that he was a sufferer from chronic insomnia. The introduction explains the origin and purpose of the book: "Nearly all the 72 problems are veritable 'Pillow Problems', having been solved in the head while lying awake at night". "My motive for publishing these problems with their mentally worked solutions is most certainly not a desire to display powers of mental calculation. I have no doubt that there are many mathematicians who could produce mentally much shorter and better solutions." "Mistakes may perchance await the penetrating glance of some critical reader, to whom the joy of discovery and the intellectual superiority which he will thus discern . . . will I hope repay to some extent the time and trouble its perusal may have cost him!"

The problems are of great variety: here are some typical ones. No. 2. In a given triangle, to place a line parallel to the base such that the portions of the sides intercepted between it and the base shall be together equal to the base. No. 14. Prove that three times the sum of three squares is also the sum of four squares. No. 5. A bag contains one counter, known to be either white or black. A white counter is put in, the bag shaken, and a counter taken out, which proves to be white. What is now the chance of drawing a white counter?

Perhaps you would like to try to solve these one night when you are lying sleepless in bed. The author recommends such problems as a cure for troubled and distracted minds which will not rest but continue to worry about some vexatious matter of the previous day's happenings.

There is, however, one problem which calls for further notice. It is referred to in the preface as his "one problem in Transcendental Probabilities—a subject on which very little has yet been done by even the most enterprising of mathematical explorers. To the casual reader it may seem abnormal or even paradoxical, but I would have such a reader ask himself candidly the question, 'Is not life itself a paradox'?"

But for this serious tone, one would have suspected the author of indulging in a little leg-pulling at the expense of his readers, since it is curious that so logical a mind as his could have overlooked the fallacy in his solution. Here are the problem and solution as Dodgson gives them.

"A bag contains 2 counters as to which nothing is known except

that each is either white or black. Ascertain their colours without taking them out of the bag."

Dodgson's answer is, "One white, one black". Now this is not paradoxical as the author claims, but nonsensical, yet he arrives at his conclusion by the following argument:

"We know that if a bag contains 3 counters, 2 being black and one white, the chance of drawing a black one is  $\frac{2}{3}$ , and that any other state of things would not give this chance."

Now the chances that the given bag contains, (1) *BB*, (2) *BW*, (3) *WW*, are respectively  $\frac{1}{4}$ ,  $\frac{1}{2}$ ,  $\frac{1}{4}$ .

Add a black counter.

Then the chances that it contains (1) *BBB*, (2) *BBW*, (3) *BWW*, are as before,  $\frac{1}{4}$ ,  $\frac{1}{2}$ ,  $\frac{1}{4}$ .

Hence the chance of now drawing a black one

$$= \frac{1}{4} \cdot 1 + \frac{1}{2} \cdot \frac{2}{3} + \frac{1}{4} \cdot \frac{1}{2} = \frac{3}{4}$$

Hence the bag now contains *BBW* (since any other state of things would not give this chance).

Hence before the black counter was added, it contained *BW*, i.e. one black and one white counter.

Q.E.F."

As I expect you would prefer to enjoy that feeling of intellectual superiority by discovering the fallacy for yourself, I will refrain from pointing it out, but will content myself with remarking that if one applies a similar argument to the case of a bag containing 3 unknown counters, black or white, one reaches the still more paradoxical conclusion that there cannot be 3 counters in the bag!

Apart from this lapse, the book shows that Dodgson possessed unusual ability in solving quite difficult problems mentally: for instance, he thought of a semi-regular dekahedron, composed of two squares in parallel planes joined by eight equilateral triangles, and calculated the volume of the solid thus obtained. The problems are not all so original as this, but the book is probably the only one of any real interest to mathematicians to-day.

A note in Dodgson's diary on 29th March, 1885, gives a list of the books he hoped to publish: it includes a paper on "Infinities and Infinitesimals", and "Plain Facts for Circle Squarers", "which is nearly completed and gives actual proof of limits 3.14158 and 3.14160". Perhaps it was never published, because Dodgson sadly realised that plain facts were things no circle squarers would have the courage to face.

To turn, in conclusion, from the books to the author himself. It would be idle to pretend that he was a great mathematical discoverer or genius. His life was spent in a busy round of tutorials and lectures at Christ Church, and judging by the pamphlets of simple formulae and notes on Euclid, it appears that he seldom had to do any advanced mathematical teaching. Nevertheless he found this elementary work absorbing both his time and his interests. His diary records from time to time discoveries of simple facts, many of them quite trivial.

31st October, 1890. "This morning, thinking over the problem of finding two squares whose sum is a square, I chanced on the theorem (which seems true, though I cannot prove it) that if  $x^2 + y^2$  is even, its half is the sum of two squares. A kindred theorem that  $2(x^2 + y^2)$  is always the sum of two squares also seems true but unprovable".

5th November. "I have now \* proved the above two theorems. Another pretty deduction from the theory of square numbers is that any number whose square is the sum of two squares is itself the sum of two squares" †.

27th September, 1897. "Dies Notandus. Discovered rule for dividing a number by 9, by mere addition and subtraction. I felt sure there must be an analogous one for 11, and found it, and proved my first rule by algebra after working about nine hours".

19th December (Sunday). "Sat up last night till 4 p.m. over a tempting problem sent me from New York : to find 3 equal rational sided rt.-angled triangles. I found two, whose sides are 20, 21, 29 and 12, 35, 37, but could not find *three*".

As a lecturer he was considered rather dull, and one may reasonably suspect that this was partly due to his meticulous care in being precise and logical at all costs, not only when dealing with his beloved Euclid, but also in algebra and arithmetic. As a tutor he was very painstaking, keeping a record of the progress made by each individual student, and providing them with printed sheets of formulae and notes, and systems for working out examples to ensure adequate practice in every type of question.

Undoubtedly his greatest claim to be considered a genius was his ability to take pains. No one could have been more devoted to his work, and his conscientiousness and thoroughness, as well as his gifts of humour and his charm of manner, entitle him to a place amongst the Immortals. The world will always remember him as the author of *Alice in Wonderland*, yet we also may acclaim him as one of the illustrious English mathematicians of whom we are rightly proud.

**The President :** I am sorry that Mr. Eperson did not refer to the statement about two and one being equal to three in the "Hunting of the Snark!" One other point occurred to me, that is, that Dodgson did not seem to be aware that every integer is expressible as the sum of four squares.

I am sure you will join with me in offering our thanks to Mr. Eperson for his very interesting paper.

$$* 2(x^2 + y^2) = (x - y)^2 + (x + y)^2.$$

† "Assume  $x, y, z$  to have no common factor.

If  $x^2 + y^2 = z^2$ , then  $y^2 = (z - x)(z + x)$ .

Then if  $y$  has a pair of factors  $\mu, \nu$ , (where  $\mu > \nu$ , and  $\nu$  may be unity),

$$z + x = \mu^2,$$

$$z - x = \nu^2,$$

$$\text{and } z = \frac{1}{2}(\mu^2 + \nu^2) = \frac{1}{2}[(\mu + \nu)^2 + (\mu - \nu)^2].$$

Q.E.D."

## A NEW ATTEMPT TO PROVE THE PARALLEL POSTULATE.

BY D. R. WARD, S.J.

PERIODICALLY until the end of time there will appear published works purporting to prove the parallel postulate, and it has fallen to the writer's lot to read what is perhaps the most recent of these.\* While the exact mathematical value of such attempts can always be gauged *a priori*, it is useful to put on record any new method followed, if only to warn future generations of parallel-postulate-provers against trying it. But such attempts are not without all value; some readers may find it a congenial occupation to spend a little time in discovering the inherent fallacy; alternatively, the task may be given to a student as a profitable and novel exercise in geometry.

For these reasons, then, the following résumé of this most recent attempt is given here. And in order that the full interest of discovering the fallacy may not be denied to the reader or the luckless student to whom he delegates the task, it has been thought better to give only the proof as the author gives it, and not to point out where the fallacy lies. The writer has, however, allowed himself to make one or two comments on certain noteworthy points.

Reduced to skeleton form, the proof proceeds by the following stages.

(1) The author first gives a definition of a parallelogram, as follows: *A parallelogram is a four-sided figure having its opposite sides and angles equal.*

The stickler for economy in definition will at once pounce on the inclusion of equal angles in this, for it is a necessary consequence of the equal sides. And since it savours of begging the whole question at issue to give such a suggestive name as *parallelogram* to a figure which is not as yet known to have any parallel properties, it might have been more fitting to have given it a provisional name until the parallel properties had been proved. But these are minor points.

(2) Parallel lines are next defined: *Parallel lines are lines that, lying in the same plane, are equidistant at equidistant points.*

This is indeed a vague definition, and the reader will suspect that it is at this point that the author lays the foundation of his fallacious reasoning.

After due consideration of the context, it was decided that this novel definition was intended to convey one or other of the two following; it is not by any means clear which.

(a) *Parallel lines are coplanar lines such that the corresponding extremities of any two equal segments, one on each, are equidistant.*

\*Euclid or Einstein, a Proof of the Parallel Theory and a Critique of Metageometry, by J. J. Callahan, President of Duquesne University. (Published by the Devin-Adair Company, New York; price \$4.50.) As the title indicates, the author is concerned with more than his attempt to prove the parallel postulate; with the rest of his work, however, this article does not deal.

(b) *Parallel lines are coplanar lines such that the corresponding extremities of two equal segments, one on each, are equidistant.*

The difference is slight but extremely important. The reader will easily verify that (a) is equivalent to the ordinary definitions of parallels, but that (b) is not. The assumption that the two are equivalent is tantamount to the assumption of the parallel postulate.

(3) Next follows an existence theorem for parallelograms as defined above; the proof is by constructing a figure satisfying the requirements, and runs as follows:

*Given a triangle, to construct a parallelogram.*

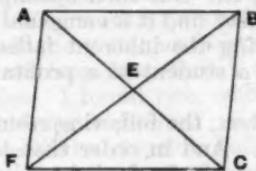


FIG. 1.

Let  $ABC$  be any triangle, and let  $E$  be the mid-point of  $AC$ . Join  $BE$  and produce it to  $F$ , making  $EF$  equal to  $BE$ .

Join  $AF$  and  $CF$ . Then  $ABCF$  is the required parallelogram.

The proof is now obvious, for the triangles  $AEB$  and  $FEC$  are congruent; similarly  $BEC$  and  $AEF$  are congruent. It follows that the opposite sides of the quadrilateral are equal. The unnecessary angle property included in the definition also follows.

It may be noted that this construction is not the simplest possible, and that the enunciation should be amplified. But there is a more serious criticism. The existence theorem for "parallelograms" should have followed immediately after the definition, for to the unlearned for whom the author writes the proof that "parallelograms" exist may suggest that the existence of parallel lines has been proved. This suggestion is strengthened by the facts that parallels have already been defined, and that there is a false implication in the word *parallelogram*.

(4) The next stage brings us to the heart of the matter; it is contained in "Proposition II", enunciated thus:

*If a transversal falling on two straight lines makes the alternate interior angles equal, any equal segments of the two lines can be made the sides of a parallelogram about the transversal as diagonal.*

Before setting out the author's proof of this theorem, it should be noted that its enunciation is faulty. The unconditioned "any equal segments" is contradicted by the condition that the given transversal shall be a diagonal. It should therefore be emended to read "... any two equal segments, one on each, terminated at their opposite extremities by the given transversal, can be made the opposite sides of a parallelogram". The clause about the diagonal is then unnecessary.

The proof is as follows :

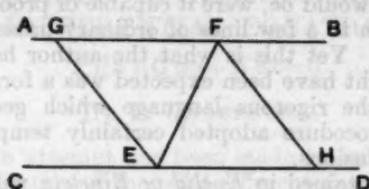


FIG. 2.

Let  $AB$  and  $CD$  be two straight lines, and  $EF$  the given transversal, such that  $\angle AFE$  and  $\angle FED$  are equal.

Let  $G$  be any point on  $AB$ , and  $H$  a point on  $CD$ , on the opposite side of the transversal to  $G$ , and such that  $EH$  and  $GF$  are equal.

Join  $GE$  and  $FH$ . Then  $GFHE$  is the required parallelogram.

The proof is once more obvious, for plainly the triangles  $GFE$  and  $HFE$  are congruent, so that  $GE$  and  $FH$  are equal.

(5) Having proved this proposition, the author goes on thus :

"We have now in two propositions, each of them clear and apodictic, completely established the basic doctrine of the parallel theory, the reality of parallel lines. They are lines that are equidistant at equidistant points.

They are equidistant not only at the extremities of the sides of the parallelogram, a thing which was proved in the first proposition, but anywhere on either line. For we may take any point  $G$  to establish the parallelism, and therefore it is proved for every point. There is only one condition required, and that is that the two lines be such that the angles formed with them by any transversal, and called the alternate-interior angles, be equal. This being given, we may take any other points on either line equidistant from the two points of intersection, and show that the pair of points in one line is equidistant from the corresponding points of the pair on the other line. In other words, the complete figure will form a parallelogram."

Now, whatever may be said to the contrary, up to the end of "Proposition II" parallel lines have not been mentioned since they were defined. Hence when the author states that he has proved the existence of parallel lines, he is not accurate ; we have to infer that what he *has* said implies in some way the existence theorem for parallel lines. Does his second proposition imply that the opposite sides of the "parallelogram" are in fact parallel, in accordance with definition (a) of parallel lines ? If it does not, there arises the subsidiary question, which is interesting enough, even though irrelevant : since he obviously imagines he has proved the existence of parallel lines, where and how has he deceived himself ?

It is almost impossible to refrain from sarcastic comment (and readers of the book will admit that the author goads one to it), but it is better to leave the whole thing exactly as it stands. One remark may, however, be permitted. It is surely unique, even in

the history of attempts to prove the parallel postulate, to relegate the proof of what would be, were it capable of proof, an enormously important theorem to a few lines of ordinary prose that is not too clearly expressed. Yet this is what the author has done. Surely the least that might have been expected was a formal enunciation, and a proof in the rigorous language which geometrical proofs demand. The procedure adopted certainly tempts one to draw uncharitable conclusions.

The attempt contained in *Euclid or Einstein* will no doubt take its place amongst the other attempts which have appeared in the last two thousand years, and amongst those of the circle-squarers and angle-trisectors; indeed it may become historical, for the method used is certainly novel. But as a method it falls very far short of the historic efforts of such men as Saccheri and Wallis; these indeed thought they had proved their case, but their fallacy lay in an unproved assumption; it was not a fallacy of elementary logic.

DUDLEY R. WARD, S.J., M.A.

### PROPOSED REPORT ON THE TEACHING OF GEOMETRY.

THE Boys' Schools Teaching Committee having nearly finished their report on Algebra are beginning to turn their attention to Geometry, and propose in the autumn to consider to what extent the 1923 report should be supplemented or amplified as regards actual teaching practice, in pre-certificate work.

Constructive suggestions would be welcomed, between now and October, especially from any teachers who have made definite experiments on points left doubtful in the 1923 report.

The following points are suggested as ones on which the Committee would welcome information:

Any experiments that have been made as to the use of the Similarity postulate to replace a postulate about parallels such as "Playfair's Axiom".

Experiments on the informal teaching of similarity *pari passu* with congruence.

Suggestions as to the working in of questions on solid geometry with the course of plane geometry.

Experience as to the passage from the deductive stage *B* to the systematizing stage *C*, and as to the extent of the latter in pre-certificate work.

Correspondence on these or other points connected with the teaching of geometry should be addressed to:

C. O. TUCKEY,

Chairman of Teaching Committee,

CHARTERHOUSE, GODALMING, SURREY.

THE METHODS OF INTEGRATION OF THE  
DIFFERENTIAL EQUATION

$$P dx + Q dy + R dz = 0.$$

BY F. UNDERWOOD.

In this article an attempt has been made to indicate the ordinary methods of integration of the total differential equation

$$P dx + Q dy + R dz = 0. \quad (1)$$

Let  $X = \frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}$ ;  $Y = \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}$ ;  $Z = \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}$ .

Then the condition for integrability of (1), i.e. the condition that (1) may possess a single integral equivalent of the form  $\phi(x, y, z) = c$ , is

$$PX + QY + RZ = 0. \quad (2)$$

For all equations considered here (2) is satisfied, and no attempt is made to discuss Pfaff's problem, when (2) is not satisfied. Of the seven methods given, the first five apply to all types of equations, but the last two only when  $P, Q, R$  are homogeneous functions of  $x, y, z$ . Probably the methods in most general use are those given in I, II and VI, but the others are interesting, and in some cases have distinct practical value as methods of integration. No attempt has been made to give the general theory of these methods, but full references are given showing where proofs and further examples may be found.\*

I. In many simple cases equation (1) is exact and may be integrated by inspection, though occasionally some rearrangement is required.

## Examples.

$$(i) \quad yz(2x+z)dx + zx(z+x)dy + xy(x+2z)dz = 0, \quad (3)$$

$$\text{i.e. } (2xyz dx + x^2z dy + x^2y dz) + (yz^2 dx + xz^2 dy + 2xyz dz) = 0,$$

$$\text{giving } x^2yz + xyz^2 = c,$$

$$\text{or } xyz(x+z) = c.$$

$$(ii) \quad (3x^2y + z + 1)dx + (x^3 + 2yz)dy + (y^2 + x)dz = 0, \quad (4)$$

$$\text{i.e. } (3x^2y dx + x^3 dy) + (2yz dy + y^2 dz) + (z dx + x dz) + dx = 0,$$

$$\text{giving } x^3y + y^2z + zx + x = c.$$

\* References are to pages in the undermentioned works, and are abbreviated as follows:

- B. : G. Boole, *Differential Equations*, 4th ed. (1877).
- F. : A. R. Forsyth, *Treatise on Differential Equations*, 6th ed. (1929).
- Fl. : A. R. Forsyth, *Theory of Differential Equations*, Part I (1890).
- G. : E. Goursat, *Cours d'Analyse Mathématique*, Vol. II, 4th ed. (1925).
- I. : E. L. Ince, *Ordinary Differential Equations* (1927).
- P. : H. T. H. Piaggio, *Treatise on Differential Equations*, revised ed. (1928).
- W. : E. B. Wilson, *Advanced Calculus* (1911).



Using these equations with (9), we find  $f(y) = y/c$ , and the integral of (9) is  $x^2y + y^2 = cz(y + z)$ .

References.—P. 138-139; F. 314-316; 572-573; B. 276-281, 291; I. 53-55; W. 255-257, 259(2).

III. Mayer's method may be illustrated by means of the following example,  $y(x^3 - z)dx + x(x^2 + z)dy + xydz = 0$ , ..... (10)

which may be written

$$dz = \frac{(z - x^2) dx}{x} - \frac{(x^2 + z) dy}{y}.$$

Since the coefficients of  $dx$  and  $dy$  are holomorphic for all finite values of  $x, y, z$  except  $x=y=0$ , we may put  $y-b=m(x-a)$ , so long as  $a$  and  $b$  are not zero, e.g. we may take  $y-1=m(x-1)$ , and (10) then gives an integral

$$z\{1 + m(x-1)\} = (m-1)x^2 - mx^3 + cx.$$

On substituting for  $m$ , this becomes

$$x^2y + yz = cx.$$

References.—G. Arts. 385, 441; P. 206-208; I. 56-57; W. 258-259.

#### IV. *The use of integrating factors.*

The function  $\mu(x, y, z)$  will be an integrating factor of equation (1), that is, (1) will become an exact equation when multiplied by  $\mu$ , if

$$-\frac{d\mu}{\mu} = \frac{Z dy - Y dz}{P} = \frac{X dz - Z dx}{Q} = \frac{Y dx - X dy}{R}, \dots \dots \dots (11)$$

where  $X, Y, Z$  are defined above.

This method may be applied in two ways:

(a) by the integration of the exact equation given after multiplication by  $\mu$ ;

(b) if a second integrating factor  $\mu_1$  can be found from (11) such that  $\mu/\mu_1$  is not necessarily constant, then  $\mu=c\mu_1$  is an integral of (1).

The two methods of application may be illustrated by the following examples :

(i) When applied to (10), the method gives for equation (11),

$$-\frac{d\mu}{\mu} = \frac{-2(x^2 + z)dy - 2ydz}{y(x^2 - z)} = \frac{2dx}{x}.$$

Hence  $\mu = x^{-2}$  and (10) becomes

$$(y \, dx + x \, dy) + \left( \frac{z}{x} \, dy + \frac{y}{x} \, dz - \frac{yz}{x^2} \, dx \right) = 0,$$

i.e.  $xy + \frac{yz}{x} = c$ , or  $x^2y + yz = cx$ , as before.

(ii) The form of equation (11) deduced from (9) is

$$-\frac{d\mu}{\mu} = \frac{2z\,dy + 2(y+2z)\,dz}{z(y+z)} = \frac{2(x^2+3y^2)(y+2z)\,dz - 4xyz\,dx}{z(x^2z+2y^3+3y^2z)} \\ = \frac{4xy\,dx + 2(x^2+3y^2)\,dy}{y(x^2+y^2)}.$$

Hence two different integrating factors of (9) are

$$\mu \equiv (yz + z^2)^{-1/2} \quad \text{and} \quad \mu_1 \equiv (x^2y + y^3)^{-1/2}.$$

Either of these may be used to make (9) exact, and the integral  $x^2y + y^3 = c(yz + z^2)$  follows immediately, but the same integral is also obtained by taking the square root after putting  $\mu = c^2\mu_1$ .\*

Reference.—Fig. 14-18.

## V. Bertrand's method.

If  $\alpha(x, y, z) = a$  and  $\beta(x, y, z) = b$  are any two independent integrals of the equations

then  $f(a, \beta)$  can be found such that (1) can be written as

where

$$A : B = \frac{\partial f}{\partial \alpha} : \frac{\partial f}{\partial \beta},$$

and  $f(a, \beta) = c$  is the integral of (1).

### Examples.

(i) For equation (10), the subsidiary equations (12) become

$$\frac{dx}{0} = \frac{dy}{2y} = \frac{dz}{-2(x^2 + z)}.$$

A first integral is  $x=a$ , and (using this) a second integral is

$$y(z+a^2)=b, \text{ i.e. } y(z+x^2)=b.$$

Hence, taking  $a=x$ ,  $\beta=x^2y+yz$ , the equation

$$A \, dx + B \{ 2xy \, dx + (x^2 + z) \, dy + y \, dz \} = 0$$

must be identical with (10); hence

$$-\beta da + a d\beta = 0.$$

giving  $\beta \equiv c\eta$ , or  $x^2y + yz \equiv cx$ , as before.

(ii) For equation (9) the subsidiary equations (12) are

$$\frac{dx}{2(x^3+3y^3)(y+2z)} = \frac{dy}{-4xy(y+2z)} = \frac{dz}{4xyz}.$$

\* See also the footnote to section VI.

Hence  $2xy dx + (x^2 + 3y^2) dy = 0$ , giving  $a = x^2 y + y^3$ , and therefore  
and  $z dy + (y + 2z) dz = 0$ , giving  $\beta = yz + z^2$ .

It is found immediately that the equation  $A da + B d\beta = 0$  is identical with (9), if

$$A : B = yz + z^2 : -(x^2 y + y^3) = \beta : -a,$$

so that (9) becomes  $\beta da - a d\beta = 0$ ,

giving  $a = c\beta$ , or  $x^2 y + y^3 = c(yz + z^2)$ , as before.

References.—F. 331-332; Fl. 18-20; P. 145; I. 59-60; G. 595-597.

### VI. An integrating factor for homogeneous equations.

When the functions  $P, Q, R$  are homogeneous in  $x, y, z$  an integrating factor of (1) is

$$\lambda = (Px + Qy + Rz)^{-1},$$

provided that  $(Px + Qy + Rz)$  is not identically zero. It is obvious that this method is of great practical utility, for, save for the exceptional case noted above, the integrating factor can be written down without any preliminary investigation, the equation is thereby rendered exact, and can then be integrated.

Thus, for equation (9),

$$\lambda = \{yz(y+z)(x^2+y^2)\}^{-1},$$

and the equation becomes

$$\frac{2x dx}{x^2+y^2} + \frac{\{(y+z)(x^2+3y^2) - y(x^2+y^2)\} dy}{y(y+z)(x^2+y^2)} - \frac{(y+2z) dz}{z(y+z)} = 0,$$

$$\text{or } \frac{2x dx + 2y dy}{x^2+y^2} + \frac{dy}{y} - \frac{dz}{z} - \frac{dy+dz}{y+z} = 0,$$

giving  $x^2 y + y^3 = c(yz + z^2)$ , as above.\*

In the exceptional case when  $1/\lambda = 0$ , this method may be replaced by one of the earlier methods, or by that given in VII. Thus for the equation

$$(y^2 + z^2 - zx - xy) dx + (z^2 + x^2 - xy - yz) dy + (x^2 + y^2 - yz - zx) dz = 0, \quad (14)$$

$1/\lambda = 0$ , but method IV or VII gives the integral

$$x^2 + y^2 + z^2 = c(x + y + z)^2.$$

The method discussed in this section applies to the more general equation in  $n$  variables,

$$P_1 dx_1 + P_2 dx_2 + \dots + P_n dx_n = 0,$$

\* Though not usually desirable as a practical method of integration, after  $\lambda$  has been found, instead of proceeding to the final integration, this method may be used in conjunction with method IV. Thus, if  $\mu$  and  $\mu_1$  have the values given in IV (ii), the above integral may be written in either of the forms

$$\mu = c\lambda \quad \text{or} \quad \lambda = c\mu_1.$$

provided that this has a single integral equivalent, for an integrating factor is

$$\lambda = (P_1 x_1 + P_2 x_2 + \dots + P_n x_n)^{-1},$$

provided that  $1/\lambda$  does not vanish identically.

References.—P. 205-206; F1. 35 (Ex. 7).

### VII. Another method for homogeneous equations.

The substitution  $y=ux$ ,  $z=vx$  reduces (1) to the form

$$(P + Qu + Rv) dx + xQ du + xR dv = 0, \quad \dots \dots \dots \quad (15)$$

or, if  $P + Qu + Rv \neq 0$ , to the form

$$\frac{dx}{x} + \frac{Q du + R dv}{P + Qu + Rv} = 0. \quad \dots \dots \dots \quad (16)$$

If  $P$ ,  $Q$ ,  $R$  are homogeneous functions of  $x$ ,  $y$ ,  $z$  of degree  $n$ , after division by  $x^{n+1}$  all terms in (15), except  $dx/x$ , may be expressed as functions of  $u$  and  $v$  only. Further, if  $P + Qu + Rv \neq 0$ , (16) must be an exact equation provided that condition (2) is satisfied, while if  $P + Qu + Rv \equiv 0$ , (15) reduces to  $Q du + R dv = 0$ , which may always be integrated by means of a suitable integrating factor. It will be noted that the latter case is that in which the method VI fails, so a convenient example is furnished by equation (14). In this case equation (15) becomes, after division by  $x^3$ ,

$$(v^2 + 1 - u - uv) du + (1 + u^2 - uv - v) dv = 0,$$

$$\text{i.e. } \{u^2 + v^2 + 1 - u(u + v + 1)\} du + \{u^2 + v^2 + 1 - v(u + v + 1)\} dv = 0,$$

$$\text{or } \frac{du + dv}{u + v + 1} - \frac{u du + v dv}{u^2 + v^2 + 1} = 0,$$

$$\text{so that } u^2 + v^2 + 1 = c(u + v + 1)^2,$$

$$\text{or } x^2 + y^2 + z^2 = c(x + y + z)^2, \text{ as before.}$$

For an example of the ordinary type leading to (16), equation (9) may be used again, giving

$$\frac{dx}{x} + \frac{v(v + 2u^2 + 3u^2v) du - u(1 + u^2)(u + 2v) dv}{uv(u + v)(1 + u^2)} = 0,$$

$$\text{i.e. } \frac{dx}{x} + \frac{v(1 + u^2) + 2u^2(u + v) du}{u(u + v)(1 + u^2)} - \frac{(u + 2v) dv}{v(u + v)} = 0,$$

$$\text{or } \frac{dx}{x} + \frac{2u du}{1 + u^2} + \frac{du}{u} - \frac{dv}{v} - \frac{du + dv}{u + v} = 0,$$

$$\text{giving } xu(1 + u^2) = cv(u + v),$$

$$\text{or } x^2y + y^2 = cz(y + z), \text{ as before.}$$

References.—B. 281-282; P. 144; W. 259 (3).

*Conclusion.* It is rather difficult to compare and contrast these seven methods with regard to either theoretical elegance or practical utility, but perhaps one may venture to note how they stand with respect to the following tests :

(a) How many integrations are required ?  
 (b) Is the method general (that is, available for all types of equations) or special (i.e. limited to certain equations only) ?

The classification is then as follows :

Method	I	II	III	IV	V	VI	VII
Number of integrations	1	2	1	2	3	1	1
General or Special	S.	G.	G.	G.	S.	S.	S.

As pointed out above, methods VI and VII can be applied to homogeneous equations only, but are probably the best for such equations, for method VI is infallible (except when  $Px + Qy + Rz = 0$ ), and no manipulation or "guessing" of integrals is required, while method I can be used only for comparatively simple equations. Thus, on theoretical grounds, Mayer's method (III), being general and requiring only one integration, should be placed first, but it must be remarked that, for some equations, this single integration may prove more troublesome than the two or three integrations required by other methods. It is doubtful if in actual practice (apart from homogeneous equations) any other method is used so frequently as that given in II.

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912. *Mathematical books belonging to Jonathan Swift.* The Cambridge University Press has just (1932) published, in a limited edition, an account by Harold Williams of Dean Swift's library, with a facsimile of the original sale catalogue. The mathematical books included are only three in number, the second edition (Cambridge, 1713) of Newton's *Principia*, Joseph Moxon's *Tutor to Astronomy and Geography* (4th edition, London, 1686) and *Mechanick Exercises* (no date given, the 1st edition appeared 1677-83) by the same author. The last-mentioned book is included in a list of five works recorded in a manuscript "Catalogue of Books belonging to Dr. Swift taken about Octbr 6th 1742", now at Abbotsford, as being missing from the library. The entry in question (Williams, *loc. cit.* p. 12) is "Dr Corbet has Moxon's Mechanical Exercises"; the Rev. Francis Corbet was one of Stella's executors and a successor of Swift in the Deanery of St. Patrick's, Dublin.

A fourth book in the sale catalogue, "451 Beaumont's mathematical Slaying Tables, Dublin 1712", can scarcely be included among mathematical works; the full title, as given in the British Museum Catalogue, is "Mathematical Slaying Tables, or the great and only Mystery of weaving linen cloth explain'd. To which is added an abstract of all the statutes relating to the linen manufacture. By Joseph Beaumont, merchant". According to the *N.E.D.* (s.v. *Slaying*, vbl. *ab.* 2), "sleying" is "the proportioning of the counts of warp to the different sets of slay, so as to preserve a uniformity of fibre in similar species of cloth". [Per Mr. F. P. White.]

913. On étouffe l'esprit des enfans sous un amas de connaissances inutiles ; mais de toutes les sciences la plus absurde, à mon avis, et celle qui est la plus capable d'étouffer toute espèce de génie, c'est la géométrie. Cette science ridicule a pour objet des surfaces, des lignes et des points qui n'existent pas dans la nature. On fait passer en esprit cent mille lignes courbes entre un cercle et une ligne droite qui le touche, quoique dans la réalité on n'y puisse passer un fétu. La géométrie, en vérité, n'est qu'une mauvaise plaisanterie.— Voltaire, *Jeannot et Colin*. [Per Mr. J. B. Bretherton.]

## THE ACCURACY OF FIGURES. FURTHER REMARKS.

BY H. BERRY.

CONSIDERING the pertinent remarks on "Approximations" in *The Teaching of Arithmetic in Schools*, the steady change of treatment of the subject in text-books and the recent article by Mr. Inman in the *Gazette*, it seems that considerable independent investigation of this matter has been going on. Also few people will disagree with Mr. Inman when he says that the teaching of the subject leaves room for improvement. That here is matter worth spending time on in the classroom is clear—but the same may be said of many more topics than we can ever have time to deal with. It is in the belief that the subject of the accuracy of figures, or approximations, is one we really ought not to omit that I venture further remarks.

The discussion is about the necessity of crossing out figures at the end of some calculations and the possibility of crossing out figures at the beginning of others. The object of the discussion is to find, if possible, rules for each of the two cases, accurate enough and yet simple enough for class use. For the operations of multiplication and division, rules have recently been printed in the *Gazette* which are certainly simple enough and which as regards sufficient accuracy are so far unchallenged. I should like to comment further on them.

To establish the rules before a class, multiplications and divisions, in which one puts a ring round any doubtful figure and considers the possible error in each, are most convincing in themselves. Maximum and minimum possible products (or quotients) complete the introduction; it only remains to apply the rules when occasion arises.

It is, however, interesting to study the weakness of the rule which says "give as many figures in the answer as in the least accurate of the data" by using the identity :

$$(x + \epsilon)(y + \epsilon) = xy + \epsilon(x + y) + \epsilon^2.$$

$x$  and  $y$  are to be two numbers each with possible error  $\epsilon$ . It will be sufficient to take  $x$  and  $y$  as integers correct to the nearest unit, i.e.  $\epsilon = \pm \frac{1}{2}$ . We may neglect  $\epsilon^2$  and study  $\pm \frac{1}{2}(x + y)$  in relation to  $xy$ .

Now consider  $x \simeq 120$ ,  $y \simeq 90$ ,

i.e.  $xy \simeq 10,000$ , and  $x + y \simeq 200$ ;

hence  $\epsilon(x + y) \simeq \pm 100$ .

Having decided to work with integers we must retain zeros to indicate the number of significant figures considered. This example considers the case of a small 3-figure number multiplying a large 2-figure number. The maximum possible error is about  $\pm 1$  in the 3rd figure of the answer and thus a 3-figure answer would extract more information from the figures available than the 2-figures suggested by the rule. Had the product, through  $x$  being a little smaller say, been just less than 10,000, i.e. beginning with a 9, then a 2-figure answer would be satisfactory, having a possible error of about 1 in its 2nd figure. Of course there is little difference between a relative error of 1 in 102, for example, and 1 in 98. It is in reading

off the 3-figure number to 2 figures that accuracy may be lost. In all cases where the product begins with a small figure the rule is liable to be weak through not giving enough figures. As different examples are investigated it becomes more and more apparent that "bad"  $(n+1)$ -figure accuracy may be better than "correct to  $n$  figures" accuracy, and that, in deciding to how many figures an answer should be given, it is often the possible absolute error and particular figures of the original product which matter rather than broad considerations of relative error.

If zeros are added to  $x$  to consider the cases of greater difference in the number of significant figures in the numbers to be multiplied, the relative error is controlled by that of  $y$  and settles down to about 1 in 200, for which a 4-figure answer may occasionally be preferable to any other. A 3-figure answer will usually be best. If  $x$  is reduced to make the product begin with 9 never more than 3 figures should be given and the 2 figures of the rule will usually be adequate.

The example of  $x, y$  taken as 120, 90 typifies the cases where the rule will be seen at its worst. This is because the product begins with 1 and the smaller of the numbers multiplied begins with 9, thus having the minimum relative error if it is not to have more significant figures. Cases of which 32, 32 is a type should often have an extra figure in the answer. For this example  $xy \approx 1000$ ,  $\epsilon(x+y) \approx \pm 30$ . The first example is more extreme.

The rule never fails by giving completely useless figures (if  $\epsilon = \pm \frac{1}{2}$ ). An example of type 90, 10 is an extreme test of this. Its failure in giving too few figures is more serious if we remember that all the time we have discussed maximum errors where such errors as exist may be compensating. On the other hand we have taken  $\epsilon = \pm \frac{1}{2}$  although we may want to use the rule as a guide when  $\epsilon$  is much greater. For most carefully drawn, simple compass constructions, from which measurements to 2 decimal places are taken, we may perhaps consider  $\epsilon = \pm 1$ . With a poker-pointed pencil, chewed ruler and waggly compasses  $\epsilon$  may be much larger. For drawings of any difficulty, especially if a protractor is used,  $\epsilon$  must be greater than 1. As  $|\epsilon|$  increases from  $\frac{1}{2}$  the rule begins to have weaknesses in sometimes giving too many figures, but less often does it give too few. It seems reasonable to adopt it as a rule to be borne in mind in all work with  $\pi$ , surds, logarithms and practical measurements. The companion rule "work with one more figure than is required in the answer" follows and is remarkably reliable.

The cases of division in general and multiplication of more than two numbers may be discussed from separate identities but more or less follow from the single multiplication already discussed. Addition is of course a separate matter.

May I add that Schneider Cup and altitude judges and all who time speed-boats and cars are aware of the limitations of figures. The journalists, after their now notorious error of 11th March, 1929, in giving Seagrave's time as 15.56 secs. and average speed as 231.36246 mls./hr., are usually content with a 4-figure result. We must be as precise as possible in the schools.

H. BERRY.

ON STIRLING'S THEOREM AS A DEFINITION OF THE GAMMA FUNCTION.

BY REV. M. F. EGAN.

In the following note, Stirling's formula for a factorial is first found by a very simple process. The expression so obtained has a meaning for non-integral as well as integral values of the variable. The function it defines is easily seen (§§ 3 ff.) to possess the fundamental properties of the Gamma function, and provides an easy and natural avenue of approach to its study.

*A First Approximation to  $\log(n-1)!$*

1. Since  $\log x$  increases with  $x$ , we have

$$\log x < \int_x^{x+1} \log \xi d\xi < \log(x+1).$$

Putting  $x = 1, 2, \dots, n-1$  successively and adding, we get

$$\log(n-1)! < \int_1^n \log \xi d\xi < \log n!$$

The difference of the extreme members is  $\log n$ , and the value of the integral is  $n \log n - n + 1$ ; we therefore get, ( $\theta$  being a positive number less than unity),

$$(1) \dots \quad \log(n-1)! = (n-\theta) \log n - n + 1.$$

This approximation is often useful in problems dealing with large numbers.

*Stirling's Formula for  $\log(n-1)!$*

2. The integral

$$A(x) = \int_x^{x+1} \log \xi d\xi,$$

figuring in the preceding paragraph, is the area of the curve  $y = \log z$  between the ordinates at  $x$  and  $x+1$ . We obtained the limits  $\log z$  and  $\log(x+1)$  for this area by replacing the arc successively by parallels to the  $x$ -axis through the extremities of the arc. We got a closer approximation by replacing the arc by the chord. The area of the trapezium so obtained is  $\frac{1}{2} \log x + \frac{1}{2} \log(x+1)$ , and it is clearly less than  $A(x)$ , since the curve is concave downwards. We can write

$$(2) \dots \quad A(x) = \frac{1}{2} \log x + \frac{1}{2} \log(x+1) + \Delta(x),$$

where  $\Delta(x)$ , the area between the chord and the arc, is positive. Putting  $x = 1, 2, \dots$  and summing as before, we get

$$n \log n - n + 1 = \log(n-1)! + \frac{1}{2} \log n + \sum_{x=1}^{n-1} \Delta(x).$$

In order to make any use of this formula, we want to estimate the value of its last term, at least approximately. To do so, we shall prove that the series

$$(2.1) \dots \quad \phi(x) = \Delta(x) + \Delta(x+1) + \Delta(x+2) + \dots$$

is convergent, positive, and less than  $1/12x$ . Assuming this for the moment, it follows that

$$\sum_{n=1}^{x-1} \Delta(x) = \phi(1) - \phi(x),$$

and we get the result

$$(2.2) \dots \log(n-1)! = (n-\frac{1}{2}) \log n - n + c + \phi(n),$$

where the constant

$$(2.3) \dots c = 1 - \phi(1)$$

lies between  $11/12$  and  $1$ , and  $\phi(n)$  is positive and less than  $1/12n$ .

We may note that from the equation (2) we get at once

$$(2.4) \dots \Delta(x) = (x + \frac{1}{2}) \log \frac{x+1}{x} - 1,$$

so that the function  $\phi(x)$  is (assuming its convergency) completely defined, as is also the constant  $c$ .

It remains to prove that  $\phi(x)$  is convergent. If in the integral giving  $A(x)$ , we replace  $\xi$  successively by  $x+t$  and  $x+1-t$ , we may put  $A(x)$  equal to half the sum of the two equal integrals so obtained, i.e.

$$A(x) = \frac{1}{2} \int_0^1 \{\log(x+t) + \log(x+1-t)\} dt.$$

To get  $\Delta(x)$ , we subtract  $\frac{1}{2} \log x + \frac{1}{2} \log(x+1)$ ; we therefore have

$$(2.5) \dots \Delta(x) = \frac{1}{2} \int_0^1 \log \frac{(x+t)(x+1-t)}{x(x+1)} dt$$

$$= \frac{1}{2} \int_0^1 \log \left\{ 1 + \frac{t(1-t)}{x(x+1)} \right\} dt.$$

This is positive and less than

$$\frac{1}{2} \int_0^1 \frac{t(1-t)}{x(x+1)} dt = \frac{1}{12} \frac{1}{x(x+1)} = \frac{1}{12} \left( \frac{1}{x} - \frac{1}{x+1} \right).$$

Substituting  $x+1, x+2, \dots$  for  $x$  and summing, it follows that

$$\phi(x) = \Delta(x) + \Delta(x+1) + \dots < \frac{1}{12x},$$

which proves our proposition.\*

### The Gamma Function.

3. The function  $\phi(x)$  is defined for all positive values of  $x$ , and the constant  $c$  is definite. Following the analogy of (2.2), we define the function  $\Gamma(x)$  by

$$(3) \dots \log \Gamma(x) = (x - \frac{1}{2}) \log x - x + c + \phi(x).$$

\* In the same way we get, by taking the next term in the logarithmic series,

$$\Delta(x) > \frac{1}{12x(x+1)} - \frac{1}{120x^2(x+1)^2},$$

and if we note that

$$x^{-3} - (x+1)^{-3} = 3x^{-3}(x+1)^{-2} + x^{-3}(x+1)^{-3} > 3/x^3(x+1)^2,$$

we find

$$\phi(x) > \frac{1}{12x} - \frac{1}{360x^3}.$$

It is easy to get another term or two of the asymptotic series in this way, with the appropriate inequality.

*Fundamental Properties.*

3.1

(a) ...  $\Gamma(1) = 1$ .

(b) ...  $x\Gamma(x) = \Gamma(x+1)$ .

We verify this by taking logarithms and noting that

$$\phi(x) = \Delta(x) + \phi(x+1).$$

(c) When  $x \rightarrow \infty$ ,

$$\Gamma(x+t) \sim x^t \Gamma(x),$$

or, more precisely, the function

(3.1) ...  $q(x, t) = \log \Gamma(x) + t \log x - \log \Gamma(x+t)$

tends uniformly to zero when  $x \rightarrow \infty$ ,  $t$  being bounded. For we find

$$q(x, t) = \phi(x) - \phi(x+t) + t - (x+t - \frac{1}{2}) \log \left(1 + \frac{t}{x}\right)$$

$$= \phi(x) - \phi(x+t) + \frac{t-t^2}{2x} - \frac{t^2}{x^2} \left\{ \left(\frac{1}{4} - \frac{t}{6}\right) - \left(\frac{1}{6} - \frac{t}{12}\right) \frac{t}{x} + \dots \right\}$$

If  $|t| < a$ , the expression in brackets is less than

$$\frac{1}{4}(1+a)(1-a/x)^{-1}.$$

We have therefore

(3.11) ...  $q(x, t) = \phi(x) - \phi(x+t) + \frac{t-t^2}{2x} + \frac{\mu t^2}{x(x-a)},$

where  $|\mu| < (1+a)/4$ . We see that  $q(x, t) \rightarrow 0$  even if  $t \rightarrow \pm \infty$  provided that  $t = o(x^{\frac{1}{2}})$ .3.2. For negative values of  $x$  we may for the present define  $\Gamma(x)$  by means of property (b) of the preceding paragraph. For real values of the variable, the three properties (a), (b), and (c) may be taken to define the function. The advantages of our method of approach are that we have proved that a function exists having these properties, and that it satisfies Stirling's formula; also, as we shall see later, we can apply our first definition to the case of complex variable. We may note here that for real positive values of  $x$ ,  $\Gamma(x)$  is always positive, since its logarithm is real.*Other Properties of the Gamma Function.*

## 4. The product-formula,

(4) ... 
$$\Gamma(t) = \lim_{n \rightarrow \infty} \frac{n^t (n-1)!}{t(t+1) \dots (t+n-1)},$$

follows at once from the fundamental properties, since the expression on the right is equal to

$$n^t \Gamma(n) \Gamma(t) / \Gamma(n+t),$$

and

$$n^t \Gamma(n) / \Gamma(n+t) \rightarrow 1.$$

4.1. Again, the result that (for  $t$  positive)

(4.1) ... 
$$\Gamma(t) = \lim_{x \rightarrow \infty} x^t B(x, t),$$

where

$$B(x, t) = \int_0^1 u^{x-1} (1-u)^{t-1} du = \int_0^\infty (1+v)^{-x-t} v^{t-1} dv$$

follows equally easily. First, suppose  $x$  an integer. Integration by parts shows that in this case

$$B(x, t) = \Gamma(x)\Gamma(t)/\Gamma(x+t),$$

and the limit sought is the same as that just found. Next, suppose that  $x$  lies between the consecutive integers  $n$  and  $n+1$ . When  $x$  increases,  $x^t$  increases and  $B(x, t)$  diminishes; hence the required limit will lie between the limits of

$$n^t B(n+1, t) \quad \text{and} \quad (n+1)^t B(n, t)$$

when  $n$  tends to infinity. But both these limits are equal to  $\Gamma(t)$ .

It is useful to note that if  $a$  and  $b$  are bounded when  $x \rightarrow \infty$ ,  $\Gamma(t)$  is the limit of  $(x+a)^t B(x+b, t)$ , provided that  $b$  is real; for  $(x+a)^t/(x+b)^t \rightarrow 1$ .

4.2. To prove that, for all positive values of  $x$  and  $t$ ,

$$(4.2) \dots \quad B(x, t) = \Gamma(x)\Gamma(t)/\Gamma(x+t).$$

Let  $R(x, t)$  denote the ratio of the two expressions. When  $x \rightarrow \infty$ ,  $x^t B(x, t) \rightarrow \Gamma(t)$  and  $x^t \Gamma(x)/\Gamma(x+t)$  tends to unity, hence  $R(x, t)$  tends to unity.

Again, it is easily verified that

$$R(x, t) = R(x+1, t) = \dots = R(x+n, t)$$

$$= \lim_{n \rightarrow \infty} R(x+n, t) = 1,$$

which proves the proposition.

*The Gamma-Integral.*

4.3. Since

$$x^t B(x+1, t) = \int_0^\infty \left(1 - \frac{v}{x}\right)^x v^{t-1} dv,$$

we conjecture that its limit  $\Gamma(t)$  is equal to

$$G(t) = \int_0^\infty e^{-v} v^{t-1} dv.$$

We can take  $G(t)$  as the limit for  $x$  infinite of

$$G(x, t) = \int_0^x e^{-v} v^{t-1} dv = x^t \int_0^1 e^{-u} u^{t-1} du.$$

Now, since  $e^{-u} > 1 - u$ ,

$$G(x, t) > x^t \int_0^1 (1-u)^x u^{t-1} du = x^t B(x+1, t).$$

Again, since  $e^{-u} < 1/(1+u)$ ,

$$G(x, t) < x^t \int_0^1 (1+u)^{-x} u^{t-1} du$$

$$< x^t \int_0^\infty (1+u)^{-x} u^{t-1} du = x^t B(x-t, t).$$

When we make  $x \rightarrow \infty$ , these two inequalities prove that

$$G(t) = \Gamma(t).$$

*Properties of the function  $q(x, t)$ .*

4.4. We find easily

$$q(x, t) - q(x+1, t) = \log(x+t) - (1-t) \log x - t \log(x+1).$$

If we consider the arc and the chord joining the points  $x, x+1$  on the graph of  $\log x$ , this expression gives the height of the arc above the chord at the point of abscissa  $x+1$ . Since the curve is concave downwards, the expression is positive for values of  $t$  between 0 and 1, and negative for all other values. The same holds for every term of the series

$$\sum_{r=0}^{\infty} \{q(x+r, t) - q(x+r+1, t)\} = q(x, t),$$

and consequently for  $q(x, t)$ .

Since  $\Delta(x)$  is the area between the chord and the arc, we have

$$\Delta(x) = \int_0^1 \{q(x, t) - q(x+1, t)\} dt.$$

The series for  $q(x, t)$  converges uniformly, since its remainder-term  $q(x+n, t)$  tends uniformly to zero with  $1/n$ . We can therefore integrate it term by term, and we get

$$(4.4) \dots \int_0^1 q(x, t) dt = \sum_r \Delta(x+r) = \phi(x).$$

Again, if we consider the arc and the chord  $(x, x+1)$  on the graph of  $\log \Gamma(x)$ ,  $q(x, t)$  is the height at  $x+t$  of the chord above the arc. The area between chord and arc is therefore  $\phi(x)$ .

4.5. If in (4.4) we substitute for  $q(x, t)$  its definition, we get once Raabe's integral  $R(x)$ . Alternatively, we may follow the classic procedure and write

$$R'(x) = \frac{d}{dx} \int_x^{x+1} \log \Gamma(\xi) d\xi = \log \Gamma(x+1) - \log \Gamma(x) = \log x, \\ R(x) = x \log x - x + k.$$

On the other hand,

$$R(x) = \int_0^1 \log \Gamma(x+t) dt = \int_0^1 \{\log \Gamma(x) + t \log x - q(x, t)\} dt.$$

Integrating and comparing the two results, we find

$$k - c = \phi(x) - \int_0^1 q(x, t) dt.$$

Since the left-hand member is constant and the right-hand member tends to zero with  $1/x$ , both must be zero. Thus we get the equation (4.4) and also the value of  $k$ .

### Complex Variable.

5. If  $x$  is complex, we make the logarithms in our definition uniform by a cut along the negative half of the real axis.

To show that the series (2.1), giving  $\phi(x)$ , is convergent, we note first that the remainder after  $n$  terms is  $\phi(x+n)$ . We may therefore suppose without loss of generality that the real part of  $x$  is positive and that  $|x|$  is greater, say, than unity.

We start from the identity (2.5) and note that if  $|a| < 1$ ,

$$|\log(1+a)| < |a|/(1-|a|);$$

if  $|a| < \frac{1}{2}$ , this is less than  $\frac{1}{3}|a|$ .

$x, x+1$  lies between 0 and  $\frac{1}{4}$ , and  $|x(x+1)| > 1$ ; hence

$$|\Delta(x)| < \frac{2}{3} |x(x+1)|^{-1} \int_0^1 (t-t^2) dt = \frac{1}{9|x(x+1)|}.$$

Again, if  $x = \xi \pm i\eta$  (both  $\xi$  and  $\eta$  being positive), and  $r$  is any positive number, it is easy to see that

$$|x+r| \geq (\eta + \xi + r)/\sqrt{2};$$

since in a right-angled triangle the sum of the two sides containing the right angle does not exceed  $\sqrt{2}$  times the hypotenuse.\* It follows that

$$|\Delta(x+r)| < \frac{2}{9} \left( \frac{1}{\xi+\eta+r} - \frac{1}{\xi+\eta+r+1} \right),$$

whence †

$$(5) \dots |\phi(x)| < 2/9(\xi+\eta).$$

5.1. To deal with the case where the real part of  $x$  is negative, we consider the function  $V(x) = \phi(x) + \phi(-x)$ .

Since

$$\Delta(-x-1) = \Delta(x),$$

we find easily

$$V(x) = \sum_{r=-\infty}^{+\infty} \Delta(x+r).$$

We can evidently substitute  $x_1 = x \pm n$  for  $x$ , and choose the integer  $n$  so that the real part of  $x_1$  lies between 0 and 1. We have then

$$V(x) = V(x_1) = \phi(x_1) + \phi(1-x_1) + \Delta(-x_1),$$

and each of the three terms on the right tends to zero with  $1/\eta$ .

5.11. It is easy to get an exact expression for  $V(x)$ , by using equation (3) and the identities

$$\log \Gamma(x) + \log \Gamma(-x) = -\log x + \log(\pi \cosec \pi x),$$

$$c = \frac{1}{2} \log(2\pi).$$

We thus get

$$V(x) = -\log(2 \sin \pi x) - (x - \frac{1}{2}) \{\log x - \log(-x)\}.$$

We may suppose that  $x$  lies above the real axis,

$$(x = \xi + i\eta, \xi \pm, \eta +):$$

if not, we can change the sign of  $x$  without altering  $V(x)$ . We have then

$$\log\{x/(-x)\} = +i\pi, \text{ and } V(x) = -\log(1 - e^{2i\pi x}).$$

The modulus of  $V(x)$  is therefore  $e^{-2\pi\eta}$ , if we neglect its square and higher powers.‡

\* It follows that the condition  $|x(x+1)| > 1$  is satisfied if  $\xi + \eta > 1$ .

† If we use the lemmas

$$|\log(1+a) - a| < \frac{2}{3} |a|^2 \quad \text{for } |a| \leq \frac{1}{2},$$

$$b^{-3}(b+1)^{-3} < \frac{1}{3} \{b^{-3} - (b+1)^{-3}\} \quad \text{for } b \text{ positive,}$$

we get easily enough

$$|\phi(x) - 1/12x| < \frac{1}{12} (\xi + \eta)^{-2}.$$

‡  $e^{-2\pi\eta} < 3 \times 10^{-14}$  for  $\eta = 5$ .

If  $x$  is below the real axis, we change its sign, as has been said; so that the general formula is

$$(5.11) \dots \phi(x) + \phi(-x) = -\log(1 - e^{\pm 2ix\pi}),$$

the sign being that of the imaginary part of  $x$ .\*

On the real axis,  $\phi(x)$  has logarithmic infinities at zero and the negative integers; for the terms  $\Delta(x+n-1) + \Delta(x+n)$  in its expansion give  $-\log(x+n)$ . Except for these points,  $\phi(x)$  is everywhere finite; it tends to zero with  $1/x$  to the right of the  $\eta$ -axis, and with  $1/\eta$  everywhere.

The only modification needed in (3.11) when  $x$  and  $t$  are complex is to write  $|x|$  for  $x$  in the last of the four terms on the right. Hence  $q(x, t)$  tends uniformly to zero when  $x$  tends to infinity in any direction,  $t$  being bounded, provided that  $\phi(x)$  and  $\phi(x+t)$  both tend to zero. This requires that  $x$  at infinity should not approach within a finite distance of the negative half of the real axis.

*The Gamma-Integral for a complex argument.*

5.2. Using the notation of 4.3, we have

$$G(x, t) - x^t B(x+1, t) = x^t \int_0^1 \{e^{-xu} - (1-u)^x\} u^{t-1} du.$$

If  $x$  is real and positive and  $t=r+is$ , the factor in brackets in the integrand is real and positive, and the moduli of  $x^t$  and  $u^{t-1}$  are  $x^r$  and  $u^{r-1}$  respectively. Hence

$$\begin{aligned} |G(x, t) - x^t B(x+1, t)| &\leq x^r \int_0^1 \{e^{-xu} - (1-u)^x\} u^{r-1} du \\ &= G(x, r) - x^r B(x+1, r), \end{aligned}$$

and therefore tends to zero with  $1/x$ .

$G(t)$  is therefore the limit of  $x^t B(x, t)$  when  $x \rightarrow \infty$ . But the argument of 4.1, showing that this limit is  $\Gamma(t)$ , still holds good if  $x$  is an integer. Hence  $\Gamma(t)$  is equal to  $G(t)$ , and is therefore the limit of  $x^t B(x, t)$  when  $x \rightarrow \infty$  through any sequence of positive values. This last point is important for the next paragraph.

5.3. *To prove that*

$$B(x, t) = \Gamma(x)\Gamma(t)/\Gamma(x+t)$$

when  $x$  and  $t$  are complex, their real parts being positive.

We see, as in 4.2, that the ratio of the two expressions has the period 1, and to complete the proof we have only to show that  $(x+n)^t B(x+n, t) \rightarrow \Gamma(t)$  when the positive integer  $n \rightarrow \infty$ ; in other words, that

$$B(\xi + i\eta, t) \sim (\xi + i\eta)^{-t} \Gamma(t) \sim \xi^{-t} \Gamma(t)$$

when  $\xi \rightarrow \infty$ . In the last paragraph we saw that this is true for  $\eta = 0$ ; we have therefore to show that

$$E = \xi^t \{B(\xi, t) - B(\xi + i\eta, t)\} = \xi^t \int_0^\infty \frac{v^{t-1}}{(1+v)^{\xi+i}} \{1 - (1+v)^{-i\eta}\} dv$$

tends to zero with  $1/\xi$ . Now

$$|1 - (1+v)^{-i\eta}| = \pm 2 \sin \{\frac{1}{2}\eta \log(1+v)\} < |\eta| \log(1+v) < |\eta| v,$$

\* Cf. the results given by Lindelöf, *Calcul des Résidus* (Paris, 1905), Ch. IV.

and the moduli of the other factors are got by substituting for the exponent  $t$  its real part  $r$ . Hence

$$|E| < \xi^r |\eta| B(\xi - 1, r + 1) \sim \xi^{-1} |\eta| \Gamma(r + 1) \rightarrow 0,$$

and our proposition is proved.

It follows also that  $\Gamma(t)$  is the limit of  $x^t B(x, t)$  when  $x$  tends to infinity in any direction to the right of the  $\eta$ -axis: for

$$x^t B(x, t) = x^t \Gamma(x) \Gamma(t) / \Gamma(x + t) \rightarrow \Gamma(t).$$

*Addition-Theorem for  $q(x, t)$ .*

6. The theorem is

$$(6) \dots \sum_{r=0}^{m-1} q(x, t + r/m) - q(mx, mt) = m\phi(x) - \phi(mx).$$

If we denote the left-hand member by  $\lambda(x, t)$ , we get without difficulty

$$\lambda(x, t) = \lambda(x, t + 1/m) = \dots = \lim_{s \rightarrow \infty} \lambda(x, t + s),$$

provided that  $s$  is a multiple of  $1/m$ .

Again, we find

$$\begin{aligned} q(u, v + c) - q(v, u + c) \\ = (u + v + c - \frac{1}{2}) \log(u/v) - u + v + \phi(u) - \phi(v), \end{aligned}$$

and, by means of this,

$$\lambda(x, t) - \lambda(t, x) = m\phi(x) - \phi(mx) - m\phi(t) + \phi(mt).$$

If we make  $t \rightarrow \infty$ , this gives

$$\lim_{t \rightarrow \infty} \lambda(x, t) = m\phi(x) - \phi(mx),$$

and the comparison of the two limits proves our theorem.

6.1. If we write  $1, t-1$  for  $x, t$ , and note that

$$q(1, t-1) = -\log \Gamma(t), \quad \phi(1) = 1 - c,$$

$$q(m, mt-m) - \phi(m) = (mt - \frac{1}{2}) \log m - m + c - \log \Gamma(mt),$$

we get Gauss' multiplication-theorem for  $\Gamma(t)$ .

M. F. EGAN.

914. The mathematician's best work is art, a high and perfect art, as daring as the most secret dreams of imagination, clear and limpid. Mathematical genius and artistic genius touch each other.—G. Mittag-Leffler; quoted by Havelock Ellis, *The Dance of Life*, p. 128.

Does it not seem as if Algebra had attained to the dignity of a fine art, in which the workman has a free hand to develop his conceptions, as in a musical theme or a subject for painting? It has reached a point in which every properly-developed algebraic composition, like a skilful landscape, is expected to suggest the notion of an infinite distance lying beyond the limits of the canvas.—Sylvester in his *Theory of Reciprocants*; quoted by Havelock Ellis, *The Dance of Life*, p. 128. [Per Mr. E. G. Hogg.]

915. Quant à son esprit, c'est un des plus cultivés que nous ayons; il sait beaucoup de choses, il en a inventé quelques-unes; il n'avait pas encore deux cent cinquante ans, et il étudiait selon la coutume au collège des jésuites de sa planète, lorsqu'il devina, par la force de son esprit, plus de cinquante propositions d'Euclide. C'est dix-huit de plus que Blaise Pascal, lequel, après en avoir deviné trente-deux en se jouant, à ce que dit sa sœur, devint depuis un géomètre assez médiocre, et un fort mauvais métaphysicien.—Voltaire, *Micromégas*, ch. i. [Per Mr. J. B. Bretherton.]

## MATHEMATICAL NOTES.

1068. *Timber lengths.*

I understand that the following method is used in practice to find the total number of linear feet of any size timber in a cargo. It is, perhaps, best explained by an example. A tally book or sheet is used with the natural digits in the left-hand column in increasing order to represent lengths. The quantity of each length is then entered thus :

10 feet	-	-	-	5 lengths
11 feet	-	-	-	12 lengths
12 feet	-	-	-	8 lengths
13 feet	-	-	-	11 lengths
14 feet	-	-	-	3 lengths

Starting with the number of the longest length, make this series with the numbers in the second column :

3
14 (3 + 11)
22 (3 + 11 + 8)
34 (3 + 11 + 8 + 12)
39 (3 + 11 + 8 + 12 + 5)

Now multiply this last term by the highest missing length below (in this case 9) :

$$351 \quad (39 \times 9)$$

Number of feet	-	463
----------------	---	-----

If all of a certain length are absent, include zero in the series :

10 feet	-	-	-	5 lengths
11 feet	-	-	-	0
12 feet	-	-	-	8 lengths
13 feet	-	-	-	0
14 feet	-	-	-	3 lengths;

then the total number of feet is

3
11
11
16
144 (16 × 9)
188

J. T. WILLIAMS.

1069. *A difficult converse.*

In Note 1031 (*Gazette*, XVI, p. 200) three proofs were given of the theorem that a triangle is isosceles if the bisectors of two of its angles are equal. Since then many proofs of this result have been received ; a selection is given below. In each proof *BE* and *CF* are the bisectors of the angles *ABC* and *ACB*. Among Mr. Greenstreet's papers there was a collection of proofs of the theorem, and from this collection some references to authorship have been drawn.

1. Sent by F. C. Boon, and referred by him to Chartres, *Educational Times Reprint*, 74, p. 106.

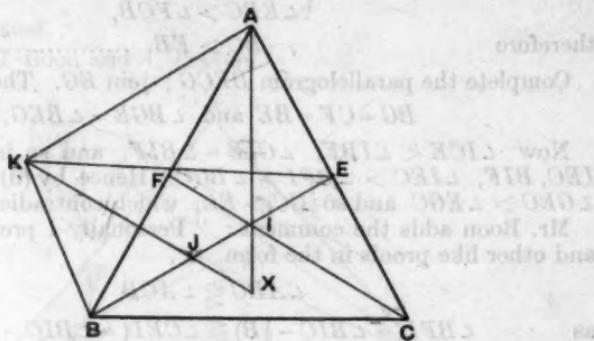


FIG. 1.

Let  $BE$ ,  $CF$  be equal, intersecting in  $I$ . Construct  $K$  so that  $KB = AF$ ,  $KE = AC$ , and let the bisector of  $\angle BKE$  meet  $BE$  in  $J$ ,  $AI$  in  $X$ . Then  $KJ = AI$ . Since  $\angle BKE = \angle BAE$ ,  $BKAE$  is cyclic, and so  $\angle AKE = \angle ABE = \frac{1}{2}B$ . Thus  $\angle JKA = \frac{1}{2}(A+B) = \angle AIE$ , and  $AKJI$  is cyclic, and since  $KJ = AI$ ,  $AK$  is parallel to  $JI$ . Thus  $\angle AKE = \angle KEB$ , or  $\frac{1}{2}B = \frac{1}{2}C$ , whence the result.

2. Sent by F. C. Boon and E. W. Burn : the former refers to it as Greenstreet's own proof.

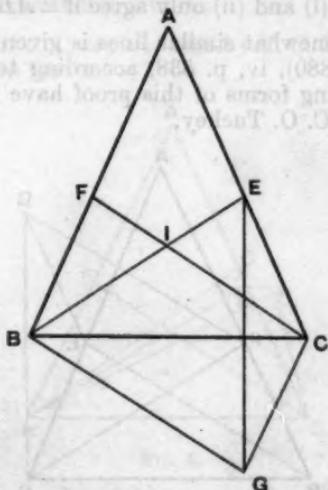


FIG. 2.

If  $\angle IBC$ ,  $\angle ICB$  are not equal, let  $\angle IBC$  be the greater. Then in the triangles  $BCE$ ,  $BFC$ ,

$$\left\{ \begin{array}{l} BE = CF, \\ BC = BC, \\ \angle EBC > \angle FCB, \end{array} \right.$$

therefore

$$EC > FB. \dots \text{(i)}$$

Complete the parallelogram  $BFCG$ ; join  $EG$ . Then

$$BG = CF = BE \text{ and } \angle BGE = \angle BEG. \dots \text{(ii)}$$

Now  $\angle ICE < \angle IBF$ ,  $\angle CIE = \angle BIF$ , and so in the triangles  $IEC$ ,  $BIF$ ,  $\angle IEC > \angle BFI > \angle BGC$ . Hence by (ii) the remainder  $\angle GEC > \angle EGC$  and so  $GC > EC$ , which contradicts (i).

Mr. Boon adds the comment: "Personally I prefer to put this and other like proofs in the form

$$\angle ABC \geq \angle ACB$$

$$\text{as } \angle BFI (= \angle BIC - \frac{1}{2}B) \geq \angle CEI (= \angle BIC - \frac{1}{2}C),$$

$$\text{or as } \angle BGC \geq \angle CEI,$$

which, since  $BG = CF = BE$ , is according as

$$\angle EGC \geq \angle GEC,$$

$$\text{or } EC \geq CG, \text{ which is as } EC \geq BF. \dots \text{(i)}$$

But in the triangles  $BEC$  and  $BFC$  we have

$$\angle EBC \geq \angle FCB \text{ as } EC \geq BF.$$

$$\text{That is, } \angle ABC \geq \angle ACB \text{ as } EC \geq BF. \dots \text{(ii)}$$

These two results (i) and (ii) only agree if  $\angle ABC = \angle ACB$ ."

3. A proof on somewhat similar lines is given by Descube, *Jour. de Math. Élém.*, (1880), iv, p. 538, according to Mr. Greenstreet's manuscript. Varying forms of this proof have been sent by F. C. Boon, C. Fox, and C. O. Tuckey.

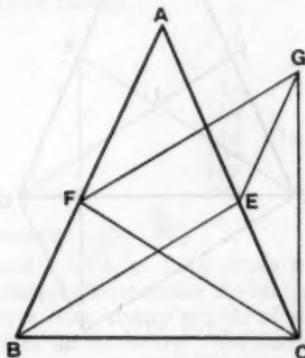


FIG. 3.

Complete the parallelogram  $BEGF$ ; join  $EG$ ,  $GC$ . Then  $\angle FGC = \angle FCG$ , and if  $B > C$ ,  $\angle FGE = \angle FBE > \angle FCE$ . Thus

$\angle EGC < \angle ECG$  and so  $EC < EG$  or  $EC < FB$ . But as in 2, if  $B > C$ ,  $EC > FB$ .

4. Casey's proof.

Sent by F. C. Boon and J. Travers.

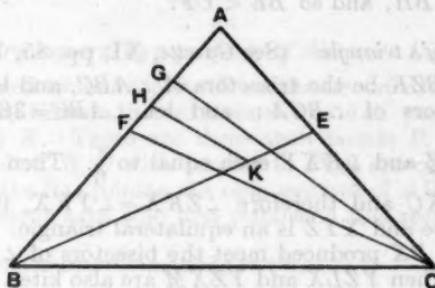


FIG. 4.

It will be sufficient to prove that if  $B < C$ , then  $BE > CF$ . Make  $\angle FCG = \frac{1}{2}B$ .

Then  $\angle BCG > B$ , and so  $BG > GC$ . Cut off  $BH$  equal to  $GC$  and make  $\angle BHK$  equal to  $\angle BGC$ . Then the triangles  $BHK, CGF$  are congruent and  $CF = BK < BE$ .

H. G. Forder points out that this proof is independent of the parallel postulate: this is logically an advantage as the result is independent of the parallel postulate.

5. A proof similar to Casey's is sent by F. C. Boon and attributed by him to E. Martin Gover.

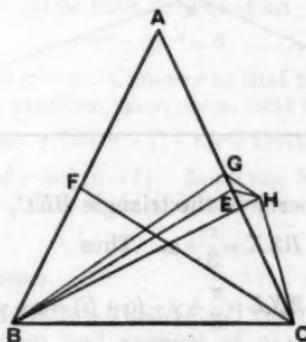


FIG. 5.

If  $B > C$ , make  $BHC$  congruent to  $BFC$ , having  $\angle BCH = B$ ,  $\angle CBH = \frac{1}{2}C$ . Then  $BH$  lies in  $\angle EBC$ . Let the circle  $BFHC$  cut  $AC$  in  $G$ . Then the order of the points  $G, E, C$  is given by  $\angle GBH = \angle GCH = B - C = 2\angle EBH$ .

Thus  $\angle BHE < \angle BHG = \angle BCG$ ,  
 $< \pi - B = \pi - \angle BCH$ ,  
 $< \angle BGH$ , since  $BCHG$  is cyclic,  
 $< \angle BEH$ .

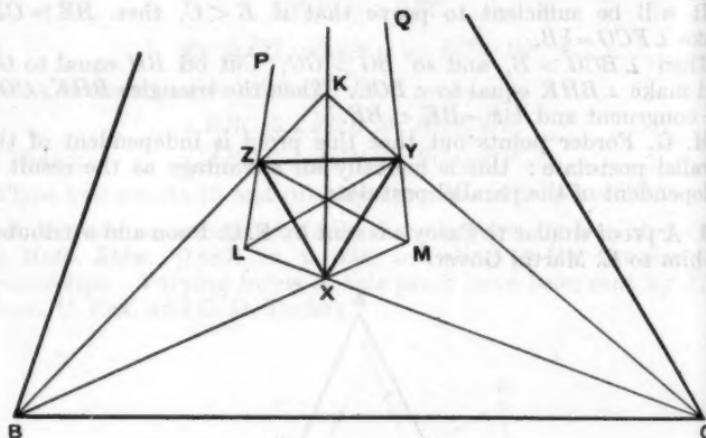
Thus  $BE < BH$ , and so  $BE < CF$ .

1070. *Morley's triangle.* (See *Gazette*, XI, pp. 85, 164, 171, 310.)

Let  $BXM$ ,  $BZK$  be the trisectors of  $\angle ABC$ , and let  $CXL$ ,  $CYK$  be the trisectors of  $\angle BCA$ : and let  $\angle ABC = 3\beta$ ,  $\angle BCA = 3\gamma$ ,  $\angle CAB = 3\alpha$ .

Make  $\angle KXZ$  and  $\angle KXY$  each equal to  $\frac{\pi}{6}$ . Then since  $X$  is the incentre of  $BKC$  and therefore  $\angle ZKX = \angle YKX$ , it follows that  $KZXY$  is a kite and  $XYZ$  is an equilateral triangle.

Let  $CX$  and  $BX$  produced meet the bisectors of  $\angle ZYX$ ,  $\angle YZI$  in  $L$  and  $M$ . Then  $YZLX$  and  $YZXM$  are also kites. Join  $LZ$  and produce to some point  $P$ ; join  $MY$  and produce to some point  $Q$ .



By the incentre property of the triangle  $BKC$ ,  $\angle KXB = \frac{\pi}{2} + \gamma$ , and so  $\angle ZXZ = \frac{\pi}{3} + \gamma$ ,  $\angle BZX = \frac{\pi}{3} + \alpha$ . Thus

$$\angle ZXZ = \frac{\pi}{3} + \gamma - (\alpha + \beta) = \alpha + \gamma,$$

and since  $YZLX$  is a kite,  $\angle LZX = \alpha + \gamma$ , and

$$\angle BZL = \left(\frac{\pi}{3} + \alpha\right) - (\alpha + \gamma) = \alpha + \beta, \quad \angle PZB = \pi - (\alpha + \beta).$$

Also  $\angle ZLX = \pi - 2(\alpha + \gamma)$  and similarly  $\angle QMX = \pi - 2(\alpha + \beta)$ . Thus  $\angle PZB = \frac{\pi}{2} + \frac{1}{2} \angle QMX$ , and as  $Z$  is the incentre of the triangle

formed by  $AB$ ,  $BM$ ,  $MQ$ , therefore  $BA$ ,  $LP$  and  $MQ$  are concurrent and  $\angle BAZ = \angle ZAY$ . Similarly  $CA$ ,  $MQ$  and  $LP$  are concurrent and  $\angle ZAY = \angle YAC$ . That is,  $ZP$  and  $YQ$  are the trisectors of the angle  $BAC$ .

Thus the points  $X$ ,  $Y$ ,  $Z$  where the trisectors meet are the angular points of an equilateral triangle.

F. C. BOON.

1071. *A query concerning a property of Simson's Line.*

"It is required to find a point  $P$  on the circumcircle of a triangle  $ABC$ , so that its Simson line with respect to  $ABC$  passes through an assigned point  $K$ . There are three such points  $P$ ,  $Q$ ,  $R$ , and the Simson lines of  $A$ ,  $B$ ,  $C$  with respect to  $PQR$  are concurrent in  $K$ , which bisects the line joining the orthocentres of  $ABC$  and  $PQR$ ".

A correspondent asks who first published this result.

1072. *Successive approximations to  $\sqrt[n]{a}$ .*

In Note 1066 a formula was given for successive approximations to  $\sqrt[n]{a}$  in the form (formula (iii) of Note 1066),

$$x_{r+1} = \frac{(n-1)x_r^{n-1} + (n+1)ax_r}{(n+1)x_r^n + (n-1)a} \dots \quad (i)$$

This has an underlying significance which is demonstrated by the following considerations. If  $\alpha$  is a root of  $f(x)=0$ ,  $x_1$  an approximation to  $\alpha$ , and if, as in Newton's rule,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)},$$

then taking, as in the previous note,  $d_1 = x_1 - \alpha$ ,  $d_2 = x_2 - \alpha$ , ..., it can be shown that in general  $d_2$  is of the order  $d_1^2$ , but if  $f''(\alpha)=0$ , then  $d_2$  is of the order  $d_1^3$ . Now take the equation

$$x^{n-r} - ax^{-r} = 0, \dots \quad (ii)$$

which is equivalent to  $x^n = a$ . Choose  $r$  so that the second derivative of the left-hand side vanishes when  $x=\alpha$ , that is, so that

$$(n-r)(n-r-1) - r(r+1) = 0.$$

The required value of  $r$  is  $\frac{1}{2}(n-1)$ . Applying Newton's formula for approximating to the roots of (ii) and taking  $r = \frac{1}{2}(n-1)$ , formula (i) will be obtained.

G. W. WARD.

1073. *Euler's Theorem.*

The usual proof of Euler's Theorem on the relation between the number of faces, edges and vertices of a simple polyhedron is obtained either by building up or taking to pieces such a surface face by face. Of these two methods the latter is preferable, as it emphasizes the fact that the condition is necessary, as the surface is already constructed, whereas it might be concluded from the former that the condition was sufficient to enable a polyhedron to be constructed, which is not the case.

The following curious example illustrates this fact.

Suppose we have a regular heptagon, a regular hexagon, a regular pentagon and four equilateral triangles, all the sides being of equal length, and we wish to make a polyhedron from them, each solid angle being a trihedral angle.

The number of faces  $F=7$ ; the number of edges

$$E=\frac{1}{2}(7+6+5+4 \cdot 3)=\frac{1}{2} \cdot 30=15;$$

the number of vertices  $V=\frac{1}{3} \cdot 30=10$ .

Hence  $F+V=17=E+2$ , and Euler's condition is fulfilled.

But it is obviously impossible to construct such a surface, as there are only 6 other faces to fit to the 7 sides of the heptagon.

H. V. MALLISON.

1074. *Note on approximations.*

This note provides an alternative to a section of Mr. Inman's article, "What is Wrong with the Teaching of Approximations?" (*Gazette*, XVI, December 1932, p. 306).

In the case of products, say  $(A \pm h)(B \pm k)$ , I suggest the following straightforward method.

Maximum limit  $AB + Ak + Bh + hk$

Minimum limit  $AB - Ak - Bh + hk$

Difference  $2Ak + 2Bh$

If all the measurements were precisely accurate the true product would be

$$AB \pm hk \pm Ak \pm Bh.$$

Now as  $h$  and  $k$  are fractional,  $hk$  is less than either  $h$  or  $k$ , so it must be omitted, and we are left with  $AB \pm Ak \pm Bh$ .

If  $h=k$ , the product is  $AB \pm h(A+B)$ .

In example (1), p. 309,  $A=2.68$  and  $B=4.12$ ,

$$h=.005, \text{ and so } h(A+B)=.005(6.8)=.034.$$

As only two places of decimals are here allowable, the correct answer is  $11.04 \pm .03$  sq. in.

*Subtraction* requires a little thought. Take for example

$$\begin{array}{r} 41.3 \pm .05 \\ 11.2 \pm .05 \\ \hline 30.1 \pm .1 \end{array}$$

If the variations are taken of the same sign, the result will be 30.1, but if of contrary sign the result will be either 30 or 30.2.

A. S. PERCIVAL.

ERNEST WILLIAM HOBSON

October 27, 1856 — April 18, 1933

## REVIEWS.

**The Fourier Integral and certain of its applications.** By NORBERT WIENER. Pp. xi, 201. 12s. 6d. 1933. (Cambridge)

The modern theory of Fourier's integral formula

$$f(x) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} g(u) e^{iux} du, \quad g(u) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(x) e^{-iux} dx,$$

starts with Plancherel's theorem, that if  $|f(x)|^2$  is integrable, then so is  $|g(x)|^2$ , and both formulae hold in the "mean-square" sense. This is an extremely elegant and satisfactory result, and it has received numerous applications in recent years. The most remarkable of these applications have undoubtedly been made by Professor Wiener.

Two of the four chapters of this book are devoted to Tauberian theorems. A Tauberian theorem is a statement of conditions under which we can pass from a limit-relation such as

$$\lim_{x \rightarrow 1} (a_0 + a_1 x + a_2 x^2 + \dots) = A$$

to an apparently more restrictive relation such as

$$\lim_{N \rightarrow \infty} (a_0 + a_1 + \dots + a_N) = A.$$

This, at any rate, is the form in which the problem was considered by Littlewood, and his famous theorem is that the deduction is possible if  $na_n$  is bounded. The connection with Fourier integrals is, to say the least of it, not apparent. It was, however, proved by Wiener not merely that Littlewood's theorem could be proved by means of Fourier integrals, but that one could establish in the same way an extremely general theorem, from which a great variety of particular cases of the above type could be deduced.

Another notable application is to the problem of prime numbers. The famous prime-number theorem is that, if  $\pi(x)$  is the number of primes less than or equal to  $x$ , then as  $x$  tends to infinity

$$\pi(x) \sim \frac{x}{\log x}.$$

All known proofs of the theorem depend partly on the theory of the Riemann zeta-function  $\zeta(s) = \sum n^{-s}$ , and partly on Tauberian arguments. It was shown by Wiener that the single fact that  $\zeta(s)$  does not vanish on the line  $\Re(s) = 1$  is all the  $\zeta$ -theory that is needed in the proof, and that the usual apparatus of complex integration and order results at infinity can be avoided.

The last chapter is on generalised harmonic analysis and almost periodic functions. This theory is due to H. Bohr, and concerns functions such as  $\cos x + \cos x\sqrt{2}$ , which are not periodic, but "almost" so. Bohr's fundamental theorem is that any such function can be represented in a certain sense by a generalised Fourier series. His original proof was very long and difficult. Afterwards a simple proof was given by Wiener, by the Fourier integral method; and here a still simpler version of the proof is given.

To have brought theorems of this degree of importance and difficulty within the scope of a single method is a remarkable achievement. We are very grateful to Professor Wiener for the clear and concise account of his theories given in this book. He is an enthusiast who does not spare himself, nor, it must be said, his readers. If you can prove Plancherel's theorem by means of Hermite polynomials, and have to learn the Hermite theory on the way, the author seems to say, so much the better!

The book is founded on a course of lectures given at Cambridge, England, in 1932. In the preface the author says that in some five years "it will be

possible to treat the Fourier Integral in a thoroughgoing and coordinate way, but for the present we shall have to content ourselves with more fragmentary treatments". It is to be hoped that the author regards this as a promise of a systematic treatise. The original literature on these subjects is already so vast that, without such treatises by acknowledged experts, the task of the young researcher will soon become almost impossible. E. C. TITCHMARSH.

**Conformal Representation.** By C. CARATHÉODORY. Pp. viii, 105. 6s. 6d. 1932. Cambridge Tracts, 28. (Cambridge University Press)

The editors of the Cambridge Mathematical Tracts are to be congratulated on securing as the writer of this monograph an authority of such eminence as Professor Carathéodory. It is long since the appreciation of a lemma associated with his name was regarded as a sign of grace in students of the subject, and now we have before us a complete exposition of the method which has advanced largely on his initiative. The tract is a translation in a modified form of lectures delivered at various times and different places and contains the theory as it has developed in this direction during the last twenty years. There is little trace of lectures in the discursive sense; the subject is treated with scholarly terseness and precision, and in the later chapters most readers would indeed welcome some looser sentences to indicate the general direction of the argument.

The first half of the tract is introductory, dealing with the Möbius transformation in terms of non-Euclidean geometry, and with elementary conformal mapping—all strictly relevant to the work which follows. Then the keynote of the method is struck by the emphasis laid on the maximum-modulus theorem in the form of Schwarz' lemma, the chapter on which opens up a field which has been little written up in English books but is well known on the Continent through the works of Bieberbach and Julia. The fundamental theorems follow with a commendably short chapter on the transformation of the boundary and a note on closed surfaces. It is in these latter chapters that the conciseness of statement is felt; indeed it is difficult to see how a reader who is not already acquainted with the doctrine of normal families of functions is to appreciate the delicacy of the argument. A historical account of theorems due to Lindelöf, Montel, Riesz, etc., would show how each step was naturally forced out of preceding steps and the artificiality of the edifice would disappear. Full references are given and a reader will do well to regard the tract as only a complete and comprehensive summary.

At the centre of the subject is the theorem of Riemann that any simply-connected domain can be conformally mapped on a circle, a result which serves not merely to indicate the possibilities of mapping but also to define an analytic function in geometric form by means of a given domain. The whole treatment depends on the method adopted in proving this theorem. The older method, introduced by Riemann himself, is intimately connected with the theory of potential and the principle of Dirichlet, and resolves itself into a discussion of boundary problems by means of Green's function and integral equations. The later method, which alone is dealt with in this tract, is shorter and more directly relevant; the ideas are taken exclusively from the theory of functions and the argument concentrates on proving that a function exists which will solve the problem, without elaborating upon the nature of the function.

The translators have done their work with care. Their version of technical terms sometimes leaves us unsatisfied, as for example, "magnification" for *Ahnlichkeitstransformation*, "in detail" for *im kleinen*, "simple" for *schlicht*, which leads to the impossible phrase "simple simply-connected domain"; on the other hand, "pricked" is almost an improvement on *punktiert* for describing an area with a point missing, and the attempt throughout to rescue terms from an over-literal translation deserves commendation. P. J.

Linear Transformations in Hilbert Space and their applications to Analysis. By M. H. STONE. Pp. viii, 622. \$6.50. 1932. American Math. Soc. Colloquium Publications, 15. (American Mathematical Society)

Special attention has of late years been directed to groups of elements with a structure resembling in a greater or less degree the structure of Euclidean space. Such a group structure constitutes an abstract space. Of the abstract spaces which might be constructed, those possessing concrete realizations are evidently logically possible and are the most apt to attract attention. Thus it is from the intimate connection of integral equations with forms in an infinite number of variables, perceived by Hilbert and expounded by him in the *Grundzüge* (1912), that there has arisen the theory of abstract space which bears his name. This space may be crudely described as a complex vector space of infinite dimensions. Elements of the space may be added, subtracted and multiplied by a complex number to yield other elements. There is, associated with any pair of elements  $f, g$ , a complex number  $(f, g)$ , and the number  $|f, g|$  which is real and positive has many of the properties of a modulus. The number  $(f - g, f - g)^{\frac{1}{2}}$  defines the "distance" between two elements and distances satisfy the "triangle inequality". The convergence of a sequence is defined in the usual way and a convergent sequence of elements has a limit. Whereas in a vector space of  $n$  dimensions there is a linear relation between any  $n + 1$  vectors, it is a postulate of Hilbert space that, whatever the integer  $m$ , there are  $m$  linearly independent elements. If  $(f, g) = 0$ , the elements  $f, g$  are said to be orthogonal: a set of elements of unit modulus and mutually orthogonal is an orthonormal set. It is possible to determine an orthonormal set  $\{\phi_r\}$  such that every element of the space can be expanded in the form

$$f = \sum_1^{\infty} a_r \phi_r.$$

Like the axes to which we refer Euclidean space this set is not unique. To illustrate these points, take as the elements of a Hilbert space those functions of a variable  $x$  whose squares are integrable, in the sense of Lebesgue, over an interval  $(a, b)$ . Addition and numerical multiplication are then defined in the usual way,  $(f, g)$  is equal to  $\int_a^b f(x) \overline{g(x)} dx$ , where  $\overline{g(x)}$  is the conjugate imaginary of  $g(x)$ , and the square of the distance between  $f, g$  is  $\int_a^b |f - g|^2 dx$ . The property that every convergent sequence has a limit becomes the Riesz-Fischer theorem on convergence in the mean; as an orthonormal set of elements we may take a set of functions normalized and orthogonal in  $(a, b)$  and the relation  $f = \sum_1^{\infty} a_r \phi_r$  is represented by the well-known expansion of a function in terms of orthogonal functions.

An operation  $T$  on an element  $f$  of Hilbert space may yield another element of that space: we then speak of the transformation  $T$  and write  $Tf = f_1$ : a transformation is linear if  $T(af + bg) = aTf + bTg$ . To pursue the Lebesgue illustration further: if

$$f_1(x) \equiv \int_a^b K(x, t) f(t) dt$$

a function of integrable square, the "kernel"  $K(x, t)$  defines a linear transformation in that particular space. Since transformations of this type are of great importance in analysis, a study of their properties on the abstract side is valuable. The following special types of transformation may be noted: (a) projections, which have properties analogous to those suggested by the name; (b) self-adjoint transformations, a sub-class of the symmetric transformations for which  $(Tf, g) = (f, Tg)$ ; (c) unitary transformations, which

preserve distance, in illustration of which we may cite the Fourier transformation

$$Tf = \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \int_0^\infty \sin xt f(t) dt.$$

The integral equations

$$If(x) = \int_a^b K(x, t) f(t) dt, \quad If(x) + g(x) = \int_a^b K(x, t) f(t) dt$$

suggest the central problem of the work reviewed, namely the study in Hilbert space of the equations

$$Tf = If, \quad Tf = If + g,$$

where  $T$  is a given transformation,  $g$  a given element,  $l$  a given number and  $f$  the unknown. The former equation admits a solution for particular values of  $l$  only : these constitute the point spectrum of  $T$ . If  $l$  is not in the point spectrum there is a transformation  $R$  such that, if  $Tf = If + g$ , then  $Rg = f$ . The behaviour of  $R$  depends upon the value of  $l$  and enables us to assign an  $l$ , not in the point spectrum, to one of three classes, the continuous spectrum, the residual spectrum or the resolvent set of  $T$ . A transformation is characterised by its spectrum. The spectrum of a self-adjoint transformation contains only real values of  $l$ , that of a unitary transformation only values of  $l$  for which  $|l|=1$ . The conformal transformations which relate the unit circle to the upper half of the plane transform the one type of spectrum into the other and the demonstration that similar relations hold between self-adjoint and unitary transformations is, though somewhat of a side-line, one of the most fascinating sections of the work which it is now my long postponed duty to review.

Though the foundations are adequately treated, the book embodies the very recent researches of the author and others : it thus provides an account of the present state of knowledge which could normally have been obtained only by a perusal of periodical literature. Such is doubtless the aim of the Colloquium publications. Topics akin to those so hastily alluded to above are discussed in all the detail which 600 pages allow and are copiously illustrated by them drawn from such widely diverse fields as continued fractions, Jacobi matrices, the boundary problems of differential equations, the Heaviside method and the Carleman theory of integral equations. The Stieltjes integral and its generalization, the Radon-Stieltjes integral, play an important part in the analysis, and the discussions of these integrals incorporated in the text should prove valuable to workers in other branches of mathematics.

Where so much is provided criticism may appear carping, but the reviewer does feel that at times the author, in a conscientious and unnecessary effort not to appear easy, has deliberately chosen an unnatural notation which conceals rather than reveals the analogies he expounds. Surely the student of abstract space may, more than the rest of us, claim the privilege of calling different things by the same name. It is thus that science progresses. In conclusion, it may be said that an introductory chapter, showing in their proper setting the main results obtained, would have provided guidance in a subject whose main trend is so frequently concealed by the details of argument and added greatly to the value of the book. For a statement of the essential properties of Hilbert space would be welcomed by many and not least by the mathematical physicist.

J. L. L.

**Knotentheorie.** By K. REINDEMEISTER. Pp. iv, 74. RM. 8.75. 1931. Ergebnisse der Mathematik, Band I, Heft 1. (Springer, Berlin)

The editors of the *Zentralblatt für Mathematik* have decided to issue a series of tracts whose object is similar to that of the French *Mémorial* series, namely to summarise the development of the various branches of mathematics during recent decades ; the earlier work has been described in the *Encyclopédie*. The plan of the undertaking is elastic, some of the articles will be in text-book

style, others mainly a summary of the literature ; the form decided on in each instance will depend on how much work has already been published in connected permanent form. This then is the first Heft of the first volume of the *Ergänzisse*.

As there is no connected account of the theory of knots in existence, this tract assumes no previous knowledge ; it deals entirely with recent advances due mainly to Alexander and to Reidemeister himself. A census of knots of lower orders was undertaken last century by Tait and Kirkman. Tait was interested because of a suggestion that atoms might ultimately be vortex rings in a hydro-dynamical ether, the difference between atoms of different elements being due to the different knottedness of the vortices. The semi-empirical methods of investigation employed by the earlier authors do not, however, lead to theorems of a broad general nature, and the development of topology, which has been such a prominent feature of mathematical investigation this century, has prepared the way for a different attack.

The main problem is to find some characteristic numbers or functions which will serve to classify knots, which will be the same for two knots of the same type, that is, for knots which can be transformed into one another without untangling, and will be different for two knots of different types. This problem has not been solved, but certain numbers and polynomials in one variable have been constructed which are the same for knots of the same type, but unfortunately not always different for knots of different types. These numbers and polynomials are connected with the group of the knot, which is the group of what remains of the space in which the knot is embedded when the knot is regarded as a singular line. The problem of determining when two such groups are isomorphic seems to be as difficult as the knot-problem itself.

The tract is packed with valuable material ; it gives the recent results and new results, concisely set out and with full proofs ; as a rule it is not difficult to follow, but the definitions of the knot-matrices, for example, are so excessively abstract that it is doubtful whether they would be intelligible to anyone who had not read the more intuitive treatment given in the papers to which reference is made.

At the end of the tract diagrams of knots up to order nine inclusive are reproduced from the paper by Alexander and Briggs, and in the text the tables of polynomials and of torsion-numbers due to these authors are given with emendations.

H. G. F.

**Linguistic Analysis of Mathematics.** By A. F. BENTLEY. Pp. xii, 315. £3.00. 1932. (The Principia Press, Bloomington, Indiana)

Each strengthening of rigour in mathematics necessitates a further refinement of notions, and a deeper digging into the foundations of the subject. Modern criticism began when such concepts as "continuous function" and "real number" were carefully analysed. It advanced another step when the fundamental properties of the natural numbers were investigated, and it has culminated in a careful consideration of fundamental logical notions such as "there is" or "implies".

The present book is a plea for pushing this movement still further ; in all work on foundations where ordinary language is used, such words as "abstract", "operation", "entity" occur, brought in from common language. These words are not usually analysed with the care that is expended on the mathematical terms proper to the subject. The author's method is to examine the language that is used by a particular worker in a particular investigation or series of investigations, to find in what way his terms are used, and to seek an interpretation which will make that use self-consistent. The problem is that of clarifying language only, avoiding psychology and philosophy.

A simple phrase, like " $x$  is a vector", is usually taken to mean that  $x$  (the mark on the paper) names or symbolises or stands for something not on the

paper, and then all the development of the theory should refer to the things named, the marks on the paper being statements about these things. But in a good many abstract treatments, the statement will be found that we may treat the symbols simply as marks on paper (and not as signs signifying anything else) and may play with these symbols in accordance with certain rules laid down, the "same symbol" being recognised as the "same" in its different contexts. These two points of view lead to different technical developments for instance, in connection with the sign of equality; for  $x = y$ , if the first view is adopted, means that  $x, y$  are names for the same thing, while, on the second view, the sign of equality is a mark on paper, and rules must be formulated before it can be used in the game. Furthermore, if the mark on paper is the name of something, this may be something physical, or something in the mind of the author or of his reader, or perhaps something else; whichever view is taken, the language used should be consistent with that view.

The author advocates no particular theory, he only asks for linguistic consistency and clarity. As an example of this method, he takes Carathéodory's proof that 0 is a number (*Vorlesungen über reelle Funktionen*, pp. 1-3); he maintains that Carathéodory shifts his ground and that his proof cannot be read consistently whichever of the above points of view is adopted. He admits that for the technical development of analysis these objections are unimportant, but maintains that they deprive the proof of all value as a contribution to foundations.

Although a page from the *Principia* is reproduced as a frontispiece, in space is devoted to Whitehead and Russell's methods than to those of Kronecker, Poincaré, Hilbert, Weyl and Brouwer. With Brouwer and with Chwistek he has little sympathy, but an unfair criticism of the latter in the earlier portion of the book is atoned for on a later page, and Chwistek's work is given due recognition.

The discussion in the book is novel and, though no definite conclusions are reached, it should be seen by everyone interested in these things. H. G. D.

**Humanism and Science.** By C. J. KEYSER. Pp. xxii, 243. 15s. 1931. (Columbia University Press, New York: Humphrey Milford)

Endless must be a thorough discussion about the essence of mathematics and its relations to Science and Life. There are so many angles from which these fascinating questions may be attacked that there could be no hope of arriving at universally acceptable conclusions about them. Yet, Mr. Keyser puts forth, with his usual enthusiasm and clarity of thought and expression, a plea for a humanistic interpretation of mathematics, which may be looked upon as a valuable contribution to the philosophy of the subject.

Mr. Keyser begins by drawing a sharp distinction between Science and Mathematics, neither of which should be considered as a branch or a division of the other, though both of them merge into "knowledge". The analysis of this distinction leads him to propose the following definitions of Science and Mathematics, which have a good deal in their favour. "Science is the enterprise having for its aim to establish Categorical propositions; in other words, it is the enterprise having for its aim to answer questions relating to the Actual World. Mathematics is the enterprise having for its aim to establish Hypothetical propositions; in other words, it is the enterprise having for its aim to answer questions relating to the world of the Possible" (p. 69). Then he goes on to describe, in turn, the relations of these two "enterprises" to Humanism, a notion which derives "its existence, its character and its power from the living sense in men, that as humans they are endowed with individual and personal autonomy; the sense, that is, that they are potentially qualified to judge, independently and for themselves, in all the great matters of human concern, and that they may hope to achieve a good life on earth by the use of their human faculties" (p. x). We give in full this description of humanism

because it helps one to understand the far-reaching implications of the final conclusion of the author, who states, at the end of this penetrating discussion, that "Every major concern among the intellectual concerns of man is a concern of Mathematics" (p. 222).

To many philosophers, this staggering conclusion will seem one-sided and open to criticism, not to mention that many others might take offence in this free granting of a superiority-complex to mathematicians. For after all, there are religious, moral, aesthetical and practical values, in the intellectual plane, which hardly demand any mathematics at all for their treatment, understanding or enjoyment. To mathematicians, however, Mr. Keyser's message will appear refreshing and encouraging. But it should not cause them to look down with contempt to those intellectual workers whose major concerns and achievements have not necessarily a mathematical flavour. In any case, the history of science might provide a neutral ground for reconciliation: that is, by allowing a Pythagorean interpretation of Mr. Keyser's pronouncements. But Pythagoras cannot be claimed exclusively either by the mathematicians, or by the philosophers, or by the religious reformers. He was a Man.

T. GREENWOOD.

**Wave Mechanics: Elementary Theory.** By J. FRENKEL. Pp. viii, 278. 20s. 1932. (Oxford; at the Clarendon Press)

Since the discovery of the New Quantum Mechanics in the years between 1924 and 1926, at least twenty text-books, and probably more, have appeared on the subject. Most of these books deal primarily with fundamental physical ideas; they show a great diversity of treatment, so that a layman, glancing at, for instance, the well-known book by Dirac, and also at the book under review, would find it hard to believe that they dealt with one and the same physical theory. It is in fact often believed that "matrix mechanics", "wave mechanics" and the mechanics of "q-numbers" are rival theories between which it is the duty of experiment to decide; whereas, in fact, they are merely different ways of stating the same theory, and all give exactly the same results when used to make predictions which can be tested by experiment. A book such as that by Dirac, which inclines to the "q-number" school, may be compared to a treatise on dynamics which starts by assuming the principle of least action, and deduces the behaviour of particles and rigid bodies from it; whereas the methods of Wave Mechanics, as used by Professor Frenkel, may be compared to a treatment which builds the dynamical theory on the basis of those properties of the motion of particles which can be proved by direct experiment (e.g. Newton's Laws of Motion).

It is hardly necessary to say that a theory such as Quantum Mechanics stands or falls by its ability to predict results in agreement with experiment. Perhaps a fairer idea of the present status of the theory can be obtained from recent books which deal with its application to restricted branches of physics, such as the book by Kronig on molecular spectra, or by Van Vleck on magnetic susceptibilities. In such books the methods of wave and matrix mechanics are both used freely, the method being chosen which will give a solution of the problem under discussion with the minimum of manipulation.

The book under review is the first of three volumes, and aims at giving a general survey of the whole subject, using only elementary mathematics. The second volume will build up the theory in a complete and rigorous way, and the third will give a systematic account of the applications of the theory to various physical phenomena. In the absence of the other two volumes, it is difficult to judge the first volume; it is, for instance, useless as a book of reference, there being no index. Probably, however, most of the physical results to which a research worker might wish to refer will be found in the two later volumes, and it is to be hoped that these will be provided with an index.

The distinctive feature of the book is the very full and illuminating discussion of each physical principle that is introduced. The discussion of the dualism between waves and particles is extremely good, as readers of Professor Frenkel's earlier (German) treatise would expect. There is an admirable and stimulating discussion of that much disputed hypothesis, the "Uncertainty Principle" of Heisenberg. Also there are very good chapters on radiation phenomena, the Exclusion Principle of Pauli, and the hypothesis of the "spinning electron". It is a book which can be thoroughly recommended to anyone with some knowledge of physics as it was in 1920, and who wishes to know something about the new ideas which have been introduced since then. I would recommend it even more strongly to anyone who has acquired a working knowledge of the *mathematics* of the new theory; a perusal of this book will stimulate him to think out for himself the physical principles which lie behind the mathematics.

N. F. MOTT.

**The Elements of Astronomy.** By D. N. MALLIK. Second edition. Pp. 234. 14s. 1931. (Cambridge)

On the first eighty pages of this work twenty-nine misprints have been counted. On the same eighty pages there are at least seven statements which may be described as misleading, questionable, ambiguous or erroneous. The style is poor and the matter certainly not so good as to atone for other deficiencies. In view of the fact that this is a second edition, the book cannot in any way be recommended.

R. O. R.

**Matriculation Trigonometry.** By C. V. DURELL. Pp. viii, 151, with tables and answers. Complete with answers, 3s. 6d.; without answers, 3s. Part I separately, 2s. 6d. and 2s. 1932. (Bell)

This book possesses the many admirable characteristics which distinguish text-books with which the author's name is associated. It is well designed, the text is written, in general, in a style adapted to the ready comprehension of pupils of the particular age and stage for which it is intended, and the exercises are plentiful and well-graded. It divides into two parts. Part I covers the syllabus required for the Northern Universities Joint Matriculation Board for "additional Mathematics". Parts I and II together contain what is needed for the Oxford and Cambridge School Certificate (additional Mathematics), for the London University Matriculation (Mathematics more advanced) and for corresponding papers of other examining Boards. The early chapters of Part I are also issued separately under the title "Stage A Trigonometry" as a short course for junior and middle forms of secondary schools.

The scheme is that now commonly adopted: definitions of tangent, sine and cosine for acute angles, exercises on their use (three- as well as two-dimensional), solutions of plane triangles by sine and cosine formulae, half-angle formulae, the cotangent, secant and cosecant, circular measure, graphs and problems. Part II begins with compound angles and manipulations derived from the formulae, and goes on to further properties of a triangle, equations, with a final chapter on mensuration. All this, with hosts of exercises, in 150 pages. As the author states in his preface, this is, indeed, a "concise course". Conciseness has its dangers, and in one or two respects the logical development is not cleanly finished off. It is true that the untidy ends would be tucked into their places by an efficient teacher, but to the unaided but intelligent reader they must remain untidy. It is the common practice nowadays to define the ratios at first for acute-angles, from right-angled triangles, and to defer the complete definitions by coordinates till a later stage—that of the "general angle". But, as in this book, an author often wants to deal with obtuse angles at a stage intermediate to these. There are two or three ways of doing this, none really satisfactory. Mr. Durell's plan is

to anticipate the coordinate definitions, applied only to angles up to  $180^\circ$ . But what happens about the angles  $0^\circ, 90^\circ, 180^\circ$ ? We find on page 15 the following: "Fig. 20 shows that as the angle  $\theta$  increases from  $0$  to  $90^\circ$ , the value of  $\sin \theta$  increases steadily from  $0$  to  $1$ ; but the value of  $\cos \theta$  decreases steadily from  $1$  to  $0$ ". (There has been no consideration of what happens at the ends of the range.) Then on page 78, "Now  $\cos \theta = 1$ ."

The fact is that until the coordinate method is used to define the ratios,  $\sin$  and  $\cos$  of  $0$  and  $90^\circ$  can only be arrived at via the concept of limiting values. The right-angled triangle shuts up at  $0$  and  $90^\circ$ . But why should the point be evaded? There should be no boy or girl who has arrived at the Trigonometry stage nowadays who has not had it carefully explained what is meant by saying that the value of the fraction  $1/x$  "tends to  $\infty$ " as  $x$  tends to  $0$ , and Mr. Durell uses this phrase in connection with  $\tan 90^\circ$  on page 94. There seems to be no good reason why he should not, round about page 30, have explained and illustrated that as  $x$  tends down to  $0$ ,  $\sin x$  tends down to  $0$  and  $\cos x$  up to  $1$ . Similarly for  $90^\circ$ , and later for  $180^\circ$ . Later, in connection with the general angle, we get the values for  $\cos 0$ , etc., included in the new definitions (by coordinates). It is the opinion of the writer of this review, that the boy of average ability can appreciate something of the idea of limiting value when introduced in this connection, while to the boy of more than average ability it is a necessary part of his training. Mr. Durell evidently does not believe that the idea of limiting value is comprehensible, even in this simple form, to the boys and girls for whom he writes. Otherwise he would not have left the vague statement on page 78, "If  $\theta^\circ$  is a small angle,  $\sin \theta^\circ \simeq \theta$ . We can test this approximation from the tables. If

$$\theta = 0.2, \quad \sin 0.2 \simeq \sin 11^\circ 28' \simeq 0.199,$$

that is nearly  $0.2$ ". There will probably be many who will go on to the study of the Calculus without reading any book on Trigonometry more advanced than this. Every teacher of Calculus knows how difficult it is to rid pupils' minds of vague ideas on approximations and to exorcise phrases like "the difference between them is infinitely small".\* The root difficulty of Calculus is to appreciate that if, as  $x$  decreases down to  $0$ , the difference between  $f(x)$  and  $x$  also decreases towards  $0$ , the ratio  $f(x)/x$  does not necessarily tend to  $1$ . It is not possible at the elementary stage to give a rigorous proof that the limiting value of  $\sin \theta^\circ/\theta$  as  $\theta$  tends to  $0$  is  $1$ , but the fact could have been correctly stated and numerically illustrated.

Somewhat unsatisfactory, too, is the treatment of inverse notation. The notation  $\sin^{-1}$  is introduced very early—on page 20. In connection with the general angle, no attempt is made to limit the definition. In fact, the idea is explicitly given than  $\sin^{-1} x$ , etc., are multiple-valued. Thus the answer to the question "what is  $\tan^{-1}(-\frac{1}{2})$ ?" is given as  $153^\circ 26'$  and  $333^\circ 26'$ . On page 113 the formula

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \left( \frac{x+y}{1-xy} \right)$$

is proved and passed over without the obvious comment, that although the three  $\tan^{-1}$  functions, as previously defined, have each two values between  $0$  and  $360^\circ$ , and two between  $0$  and  $-360^\circ$ , it is not a case of paying one's money and taking one's choice. After all, one of the chief "values" in Mathematics is that it is a training in exactness of thought and description. It is commonly agreed that in a course of any branch of Mathematics, the standard of rigour should be gradually raised. This should mean that in the early stages, assumptions may be made which will be further examined and underpinned at a later stage, and that mathematical concepts may at first be used in a limited field which will afterwards be extended. It should surely not be

\* This phrase is not used by the author of the book under review.

taken to mean that at any stage, and whatever the limitations, there should be any haziness or inexactness of definition.

The numerical part of the work is altogether admirable. The author's frequent use of the sign  $\simeq$  is a healthy reminder that all results obtained by the use of tables are necessarily approximate. (Thus  $\sin 60^\circ = \frac{1}{2}\sqrt{3} \simeq 0.866$ ).

There are many excellent things in this book to which attention cannot be called in detail, but it leaves an impression that the ship has been spoiled for lack of a ha'porth of tar. It reflects the economy of the days in which we live. Though excellently printed, a few of the pages have a confused, huddled, look, owing to the placing of two or three steps of a proof in the same horizontal line, and a few of the figures have been reduced in size below the size for comfort of eyesight. If the author could have allowed himself—or the exigencies of publication in 1932 could have allowed him—a few more pages to spread out his work and for a more leisurely treatment of some of the theoretical part, the book would have been even better than it is.

H. E. P.

**Advanced Algebra. I.** By C. V. DURELL. Pp. viii, 193, xxii. 4s. 1932.  
(Bell)

We have for so long had to content ourselves in our work with VIIth forms with text-books on algebra that are several years out of date that the appearance of Mr. Durell's book will be very widely welcomed. Mr. Durell's efforts to discard outworn methods of presentation are by now well known, and it would be difficult to overestimate the service he has rendered to the teaching of mathematics.

His latest book is an Algebra designed primarily for Higher Certificate candidates—it may be regarded either as a sequel to the same author's *New Algebra for Schools*, or as the first part of an advanced course, the second part of which is to include all the material necessary for scholarship candidates.

Passing from the purpose to the substance of the book, we are at one delighted with the freshness of presentation. In the first chapter the words "permutation" and "combination," which are now fit for little more than burial with honour, are discarded (except in the heading of the chapter) in favour of the more expressive terms "arrangement" and "selection". The treatment of the Binomial Theorem for a positive integral index follows the traditional method, but the Calculus is used in the worked examples on properties of the coefficients. Finite Series (Chapter III) are somewhat sketchily treated, the only methods shown being that of expressing  $u_r$  in the form  $v_{r+1} - v_r$ , and the method of mathematical induction. One page is devoted to a particular type of recurring series (the arithmetico-geometric series).

In the next chapter (Limits and Convergence) we enter upon the controversial stage, but here Mr. Durell has performed his task admirably. The treatment is very elementary, as is suitable in a book intended only for Higher Certificate candidates, yet it gives a clear idea of the nature of a limit. The discussion is confined to limits of a positive integral variable  $n$  as  $n \rightarrow \infty$ , being intended only as a prelude to the theory of convergence. A brief yet accurate explanation of convergence is given, embracing the comparison tests, d'Alembert's test and absolute convergence, in which connection, however, only the theorem that an absolutely convergent series is convergent is given, there being no mention of the possibility of multiplying two absolutely convergent series.

The traditional method of establishing the binomial and exponential series is discarded, and is replaced by a method depending on definite integration, being in effect equivalent to a use of Taylor's theorem. There are two advantages of this method :

- (i) it dispenses with the necessity of using Vandermonde's theorem, which always presents great difficulties to beginners;
- (ii) it can be re-applied, after the exponential theorem has been dealt with,

the meaning of  $x^n$  explained, and the formula  $\frac{d}{dx} x^n = nx^{n-1}$  established, for irrational values of  $n$ , to prove the binomial theorem for all real values of the index. But it does not appear that the author was greatly influenced by this consideration, for we find in fact no mention of irrational numbers throughout the book (except that on p. 124 it is proved that  $e$  is not rational).

The proofs given of the binomial and exponential theorems are open to the objection that they require the use of the theorem that if  $f(x)$  is positive and continuous in the interval  $(a, b)$ , where  $b > a$ , then  $\int_a^b f(x) dx > 0$ . The author is, however, careful to state that the truth of this theorem is assumed (p. 81). The chapter on the Binomial series also includes an elementary discussion of partial fractions and a section on homogeneous products.

The exponential and logarithmic series are approached via  $\int_1^t \frac{dx}{x}$ , which is defined as  $\text{hyp } (t)$ , but the identity of this function with  $\log t$  is not long delayed. The method of definite integration is employed to establish both the logarithmic and exponential series.

The last three chapters, on rational functions, theory of equations, and determinants, may be dealt with briefly. In the first of these are given elementary methods of obtaining the properties of quadratic, cubic and quartic functions and of the function  $(ax^3 + 2bx + c)/(Ax^3 + 2Bx + C)$ , the treatment being largely graphical. There is little in this chapter that a boy with mathematical ability would not have discovered for himself, but for the less gifted student it will be found invaluable. In the next chapter the elements of the theory of equations are very attractively set out: it contains sections on symmetric functions of the roots and transformation of equations; and both Newton's and Horner's methods of approximation to the solution of numerical equations are given. The last chapter contains an introduction to the theory of determinants. No generality is attempted: the student is led by easy stages from second order to fourth order determinants, the third order case being studied in some detail.

This book is really elementary and can be read by any pupil of average mathematical ability with the minimum of attention from the teacher. There are over 1000 examples, most of which will present little difficulty. The work is very attractively set out and clearly printed. No errata have been observed.

N. D.

**Elementary Calculus.** By A. S. RAMSEY. Pp. x, 178. 5s. 1932. (Cambridge)

**Elementary Coordinate Geometry.** By A. S. RAMSEY. Pp. viii, 155. 4s. 6d. 1932. (Cambridge)

(1) This is a book for beginners and in particular for those who will not become mathematical specialists. It is concise and clearly written but is probably much too difficult for the students for whom it is written—at least if they are expected to read it while still at school.

After a short introduction there are three chapters on differentiation. The first of these contains something about limits and differentials as well as the ordinary rules up to the differentiation of  $f(g(x))$ ; the treatment is algebraic, there are no diagrams, and even the application to tangents is left for the next chapter. For the kind of students concerned any proof of the fundamental limit theorems seems out of place; the reviewer is uncertain whether he himself understands the proof given of  $\lim \{f(x) + g(x)\} = \lim f(x) + \lim g(x)$ , and certainly if the corresponding theorem for a quotient is proved "similarly", it would appear that the theorem being proved is itself assumed at the step

$\lim \{f(x)/g(x)\} = \lim \{(F+\alpha)/(G+\beta)\} = F/G$ . After applications to tangents, maxima and minima, and kinematics, Chapter IV deals with the differentiation of trigonometrical and exponential functions and their inverses.

A footnote to the proof of  $\lim \{(\sin x)/x\} = 1$  quotes a proof of the formula for the area of a sector, but this proof involves the derivative of  $\sin x$ , and the differentiation of  $\sin x$  requires the limit of  $(\sin h)/h$ . Nothing is actually proved about the theory of exponentials, but it is made clear that the main results are quoted from Analysis. To the reviewer this procedure seems unsatisfactory.

The proof of  $\lim_{x \rightarrow 0^+} \{(e^x - 1)/x\} = 1$  is, as it stands, invalid, though it could easily be amended into a correct proof; it should be shown that the sum of  $1/2! + x/3! + x^2/4! + \dots$  is less than a fixed number for sufficiently small positive values of  $x$ , not that it is finite for all finite positive values of  $x$ .

The theory of inverse trigonometrical functions, including principal values, is clearly explained on pp. 56-60. But the work on p. 61 is not quite up to the standard of the bookwork; for actually  $\sqrt{1 - \cos^2 x} = \sin x$  and  $\cos^{-1}(\cos x) = x$  are not true when  $\sin x$  is negative. Also the derivative of  $\sin^{-1}(x/a)$  is given as  $1/\sqrt{a^2 - x^2}$  without qualification.

It should be said, however, that these details which have been criticized would be altogether over the heads of the students for whom the book is written, who would perhaps not have appreciated the proofs any better if the mistakes had not been there.

The capacity of the average non-specialist also seems to have been somewhat overrated in the three concise but lucid chapters on Indefinite Integrals, Definite Integrals, and Applications.

The explanation of the symbolism  $y = \int g(x) dx$  by  $dy = g(x) dx$ ,  $y = \frac{1}{d} g(x) dx$ , with  $\int$  as a convenient substitute for  $\frac{1}{d}$ , is interesting, but is not the true origin of the symbolism to be found in  $\lim S g(x) \delta x$ ?

It is mentioned incidentally that the centre of gravity of a uniform triangular lamina can be obtained "easily" without integration. What in fact is the simplest way of doing this?

The book ends with a short chapter on Partial Differentiation, which also includes a section on Small Changes.

(2) This is also a book written for non-specialists, and as these students will usually have some knowledge of Calculus, Calculus methods are rightly used when convenient; also, Chapter III deals with loci and some properties of curves derived from their equations.

Apart from these changes the book follows the lines of the older text-books up to conics referred to their axes.

There is the usual work about the straight line and circle. If it must be proved in a text-book on analytical geometry that a tangent to a circle is at right angles to the radius drawn to its point of contact, might this not be deduced from  $x dx + y dy = 0$ , instead of from the equation of the normal?

After the Circle chapter, there is an explanation of the kind of curves formed by taking sections of a cone, and there are the usual chapters on the Parabola, Ellipse and Hyperbola. Thus the student will learn three or four times over how to find the equation of the polar of  $(x_1, y_1)$  with respect to a conic and the conditions of conjugacy, etc.; but this is perhaps inevitable with the non-mathematician if he is to learn these things at all.

Determinants are mentioned, and Parametric Coordinates are used, but they are generally associated with ordinary cartesian methods. There is a short chapter at the end on polar and pedal equations.

As in the Calculus volume the style is clear, and it is possible that teachers who adopt the sequence that has been indicated may find the book useful even as an introductory course for mathematical specialists. A. R.

**Differential and Integral Calculus.** By J. H. NEALEY and J. I. TRACEY. Pp. viii, 496. 20s. 1932. (The Macmillan Company, New York)

The authors of this book have aimed at producing a text adapted alike for students in academic colleges and in engineering schools. One would therefore expect to find a skilful combination of theory and practice. And such is the case. Like, however, most idealistic aims, when translated into practice, a compromise becomes essential. Thus, in this important work, the pure mathematician will not find the utmost rigour of modern analysis, for, as the authors rightly point out, completely rigorous proofs of some of the theorems of the Calculus are quite out of place in an elementary course. Nevertheless, only proofs are given which are valid, and these often necessitate assumptions being made. Where such assumptions have to be made, they are clearly pointed out so that the student may realise some of the difficulties to be overcome.

The book opens with a comprehensive review of coordinate geometry, including a good section on graphs and curve tracing. Then follows some seven chapters on derivatives and differentiation. Maxima and minima, partial derivatives, mean value theorem, a chapter on solid geometry and some important applications are all dealt with very clearly and elegantly.

Integration next claims attention and some very thorough work is to be found in the remaining twelve chapters. The authors deal very lucidly with multiple integration, infinite series, power series and elementary differential equations. Throughout there are many valuable applications not only appropriately selected but excellently worked out.

Readers on this side of the Atlantic will notice that where we use the terms *slope* and *gradient*, the Americans speak of *inclination* and *slope*.

The whole book is indeed well planned, and the original aim very skilfully worked out. The combination of theory and practice is such that it should satisfy both the mathematical student and the engineer. Naturally the former will need to go much further, but in so doing, there will be no necessity to cast aside the principles taught in this book as being either invalid or only partially true.

A special feature should be mentioned, and that is the exceptionally clear diagrams, particularly of three-dimensional figures, many of which are quite difficult to draw. These diagrams are of immense help in leading the student to understand the intricate problem of multiple integration.

The volume is well got up and clearly printed. Each chapter concludes with a good set of exercises for the student, and answers are given after each problem. These would have been better placed, however, at the end of the book.

F. G. W. B.

**Elementary Mechanics and Hydrostatics.** By D. LARRETT and J. J. WALTON. Pp. 267. 3s. 6d., or without answers, 3s. 1932. (Harrap)

In about 230 pages of a small book the authors have contrived to include a short section on Experimental Mechanics, another on Experimental and Theoretical Hydrostatics, and a third section covering the elements of Statics and Dynamics from a theoretical point of view, the whole to form a two-years' course. The result is that the treatment of parts of the subjects is necessarily brief and rather sketchy in places. The authors have generally followed the older methods in preference to the new ones recommended in the Association's Mechanics Report. The order adopted is described in the preface as "that which has been found most successful in class teaching", but not everyone will agree with it. In particular, in part I, the lever and simple moments are only treated briefly in the latter part of a chapter on machines; while in the Theoretical Mechanics the chapter on parallel forces and moments is put very late, and friction is dealt with in the last chapter but one.

The plan of the first part (Experimental Mechanics) is to introduce any apparatus or ideas considered necessary, explaining them or allowing the

pupil to verify them wherever possible, otherwise taking them for granted; the conclusions drawn are based on the results of experiments performed. The pupil is encouraged to tabulate his results by frequent skeleton tables. The second part forms quite a good introduction to elementary Hydrostatics.

A number of points call for notice in the third part. It would have been better if the variation of  $g$  at different places had been made clearer, to explain the different values given to  $g$ . The parallelogram of forces (assumed from its experimental proof in the first part) is introduced early and resolution of forces, etc., based on it, while absolute units are also introduced early. In the chapter on centre of gravity the proof for the triangle is not sound, since the thin strips are really trapezia, not rods, and a similar objection applies to the proof for the parallelogram; the existence of a single point for the centre of gravity is more or less assumed. The last chapter contains the statements that "if  $\theta$  is small,  $\sin \theta = \theta$  and  $\cos \theta = 1$ "; it also introduces "centrifugal force".

The book is nicely got up and well printed, with plenty of diagrams and some attractive illustrations. The examples are not too numerous, but there is a collection of 150 questions, mostly from examination papers, at the end. There is also an index.

J. W. H.

**Elementary Mathematics from an advanced standpoint. Arithmetic, Algebra, Analysis.** By F. KLEIN. Translated by E. R. HEDRICK and C. A. NOBLE. Pp. ix, 274. 15s. 1932. (Macmillan)

This is a translation from the third German edition of the first volume of Klein's *Elementarmathematik*; the two appendices, *Zur Entwicklung der mathematischen Unterrichtsreform in Deutschland* and *Ergänzungen zur mathematischen und didaktischen Literatur*, have not been included. Professor Hedrick was one of the translators of Goursat's *Cours d'Analyse*; from the point of view of teachers of elementary mathematics this new venture is at least equally important. The translation is far from perfect, but is usually adequate, and though much of the fervour and enthusiasm of the original is lost, enough remains to make the advantage of being able to read the book in English of weight. The printing was done in Germany, and is a good piece of work, save for one really bad page, and a few comic errors in spelling, counting *center*, for example, as an Americanism, not an error. The price is reasonable, especially if we recall that the German edition was published at RM. 16.50.

There is now no reason for any teacher of mathematics in this country to neglect this book. It should be in every school library, in the original preferably, in the translation if need be, and anyone just beginning to teach mathematics should be persuaded or compelled to read it. The author speaks, as it were, from an eminence from which he can survey the whole domain of elementary mathematics; he can trace the main streams for us, can show us fords over what from the plain appear to be deep rivers dangerous to cross, and can point out how two seemingly different paths are really tending to the same goal.

Much of Klein's work was devoted to the elucidation of what he himself calls "the mutual connection between problems in various fields". He saw how mathematics inevitably tends to split up into narrow channels and how disastrous it would be if such a tendency were to be reflected in the teaching of elementary mathematics; no such disaster need be feared if teachers weigh and digest the ideas of his fascinating and inspiring volume.

T. A. A. B.

**Aufgabensammlung zu den gewöhnlichen und partiellen Differentialgleichungen.** By G. HOHEISEL. Pp. 148. RM. 1.62. 1933. Sammlung Göschens, 1059. (Walter de Gruyter, Berlin)

The general characteristics of the mathematical books in the *Sammlung Göschens* are probably known to English teachers. Dr. Hoheisel's compendium

of examples illustrating the theory of differential equations is remarkable, like other books in this collection, for the amount of material displayed in its 148 small pages. For though it is in some ways an appendix to the author's two earlier books on differential equations in this series, it is more than a mere collection of examples. There are three main parts: ordinary differential equations of the first order, ordinary equations of higher orders and partial equations. Each of these is split up into about half-a-dozen sections, and each section deals with some salient point of elementary theory, giving an outline of results, proofs and methods followed by fully worked examples and questions for solution, to which answers and frequent explanatory hints are given. There is also a collection of 117 miscellaneous examples on ordinary equations, with answers and remarks.

Though the exposition is extraordinarily concise, it is by no means obscure nor does it lend itself to "cramming". Even at the present (February) rate of exchange, Dr. Hoheisel gives us excellent value for money. T. A. A. B.

**Triumph der Mathematik.** By H. DÖRRIE. Pp. vii, 386. Geh. RM. 7. Geb. RM. 9. 1933. (Ferdinand Hirt, Breslau)

During recent years the firm of Hirt has published a number of books dealing with aspects of mathematics not usually treated in text-books. It was felt that it might be of interest to bring these to the notice of English readers and a request was sent to the publishers for review copies of the more recent volumes; a courteous and practical reply was received in the shape of five such books published during the last eight years. Dr. Dörrie's book is one of these; another is reviewed below, and other reviews will appear later.

Dr. Dörrie's sub-title is "One hundred famous problems from two thousand years of mathematical *Kultur*"—a better description than the more attractive phrase of the title, for the author is actually dealing with the problems, not merely talking about them. His field is restricted to elementary mathematics; that is to say, he considers problems which do not need the machinery of the calculus, though some knowledge of limits and of infinite series is desirable. But "elementary" does not necessarily mean "easy", for the first word is here sufficiently broad to allow, for example, a discussion of the law of quadratic reciprocity, of Malfatti's problem, of Steiner's three-cusped hypocycloid and of the construction of a sundial.

Although avoiding the explicit use of calculus, some sections involve the idea of the definite integral ingeniously introduced as a mean value, the mean value of  $f(x)$  being defined as

$$\mathfrak{M} f(x) = \lim_{n \rightarrow \infty} \{f(\delta) + f(2\delta) + \dots + f(n\delta)\}/n,$$

where  $n\delta = x$ . This enables the author to obtain the series for  $\sin x$  and  $\cos x$  in ascending powers of  $x$ . We may add that a vicious circle in his argument is only apparent and could easily be avoided by a slight rearrangement and an explicit justification of the inequality

$$\sin x < x < \tan x.$$

The most suitable review of this book would be a list of the problems it deals with; since this is impossible, we need only say that the collection, drawn from arithmetic, algebra, pure and algebraic geometry, solid geometry and astronomy is extraordinarily interesting and attractive. T. A. A. B.

**Lustiges und Merkwürdiges von Zahlen und Formen.** By W. LIETZMANN. 4th edition. Pp. vi, 307. RM. 8.50. 1930. (Hirt, Breslau)

This is another of the attractive publications of the firm of Hirt to which reference was made in the preceding review. Teachers who wish to collect

material of the kind described in Mr. Boon's article on "Sidetracks in Elementary Mathematics" in the February number of this volume of the *Gazette*, will find much to interest them in Dr. Lietzmann's book. The appearance of four editions in nine years is corroborative evidence for the truth of this statement. In English we have, of course, the classic work of Rouse Ball on mathematical recreations; the relative status of the present volume can perhaps be indicated by suggesting that in Rouse Ball's book the emphasis is on the adjective, in Lietzmann's on the noun; this is no disparagement of either author, but simply implies that the true bouquet of the English book is appreciated only by the educated mathematical palate, while the German one presents puzzles, jokes, catches, curiosities and amusements arising from or connected with mathematics which are yet interesting and intelligible to the possessor of a smattering of arithmetic, algebra and geometry. And Dr. Lietzmann does his job with gusto, and writes in the spirit of an aphorism of Novalis', quoted on p. 59: "Der echte Mathematiker ist Enthusiast per se". Whether he is presenting us with "The woeful ballad of the two jealous cones", or asking us to find fractions possessing a property shown in the example

$$\frac{1}{2} = \frac{2}{3} = \frac{3}{5} = \frac{5}{8} = \frac{8}{13} = \dots,$$

or pointing out geometrical forms occurring in nature and in art, or even mentioning an American play with characters including the heroine "Plain Geometry", her sister "Anna Lytic Geometry", and a college student "Cal. Q. Lus", there is never any reason to doubt the enthusiasm—we might almost say the high spirits—of the author. And since the popularity of cross-words, *Strand* puzzles and detective stories proves quite clearly that most people like a good puzzle, it is certain that a teacher of mathematics can use the material given by Dr. Lietzmann so as to lead the liking for puzzles towards an appreciation of more serious mathematics.

There are those who would contemptuously condemn an interest in what they would no doubt call the "amusing trifles" of Dr. Lietzmann's book. While not denying that the supreme appeal of mathematics lies in its austere beauty, there are two points which may be urged in defence of mathematical amusements. The first is that there are times when we may legitimately prefer an Aldwych play to *Hamlet*. The second is that occasionally mathematical pastimes have led to new realms of knowledge, as the four-colour problem and Euler's puzzle about the Königsberg bridges were the forerunners of the study of topology, perhaps the most important and striking of all recent developments in pure mathematics. We may recall the remark of a distinguished Cambridge mathematician concerning the modern theory of integration: "It all started from a fiddle-string" \*.

Save for a strikingly hideous cover the book is well got up, and the twenty plates are beautifully reproduced. The plate depicting a mathematician at the blackboard is faintly reminiscent of the Peterhouse portrait of Tait.

We congratulate Dr. Lietzmann on his jolly little book. May we ask him to see that the fifth edition contains some reference to Lewis Carroll?

T. A. A. B.

**Abstracts of Dissertations for the Degree of Doctor of Philosophy. V.**  
University of Oxford Committee for Advanced Studies. Pp. iv, 121. £1.  
1932. (Oxford; at the Clarendon Press)

These abstracts indicate the scope of recent post-graduate work at Oxford. One of the dissertations is mathematical and gives some developments of methods used by Hardy and Littlewood in the theory of numbers.

T. A. A. B.

\* *Vide* Riemann's classic paper, *Über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe.*

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